Techniques for Calculating the Efficient Frontier

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Two Fund Theorem

- The Two-Fund Theorem states that we can reach any point on the m-v frontier using only two efficient portfolios.
- Formally, let $\mathbf{X}_a$ and $\mathbf{X}_b$ be m-v portfolios with mean return R_a and $R_b \neq R_a$, respectively. Hence,
  1. A combination $\alpha\mathbf{X}_a + (1-\alpha)\mathbf{X}_b$ is an m-v portfolio.
  2. An m-v portfolio is a combination of $(\mathbf{X}_a, \mathbf{X}_b)$.
- Proof: Consider a portfolio $\mathbf{X}_z = \alpha\mathbf{X}_a + (1-\alpha)\mathbf{X}_b$, where $\mathbf{X}_a$ and $\mathbf{X}_b$ are m-v portfolios.

  We need to show that this $\mathbf{X}_z$ is a solution to m-v problem, i.e., we can write

$$\mathbf{X}_z = \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_z \end{pmatrix} \tag{1}$$

  where $R_z = \alpha R_a + (1-\alpha)R_b$ is the mean return of portfolio $\mathbf{X}_z$.

Two Fund Theorem: Proof continued

- We now have

$$\begin{aligned} \mathbf{X}_z &= \alpha \mathbf{X}_a + (1-\alpha)\mathbf{X}_b \\ &= \alpha \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_a \end{pmatrix} \\ &+ (1-\alpha)\Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_b \end{pmatrix} \\ &= \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ \alpha R_a + (1-\alpha) R_b \end{pmatrix}. \end{aligned}$$

- In addition, we know that

$$\begin{aligned} R_z &= \bar{\mathbf{R}}^T \mathbf{X}_z = \bar{\mathbf{R}}^T \left(\alpha \mathbf{X}_a + (1-\alpha)\mathbf{X}_b \right) \\ &= \alpha \bar{\mathbf{R}}^T \mathbf{X}_a + (1-\alpha)\bar{\mathbf{R}}^T \mathbf{X}_b = \alpha R_a + (1-\alpha) R_b \end{aligned}$$

- Therefore, we can write

$$\mathbf{X}_z = \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_z \end{pmatrix}. \qquad (2)$$

Two Fund Theorem: Proof continued

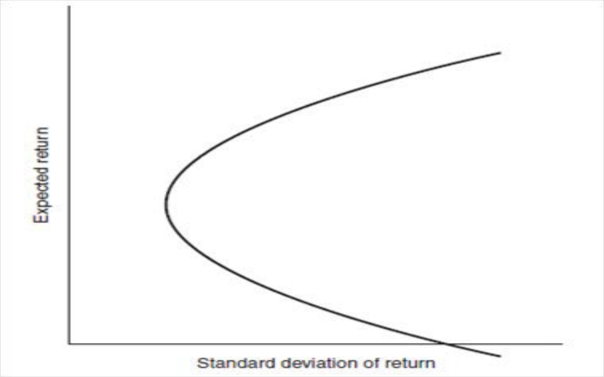
- We just showed that $\mathbf{X}_z$ is an m-v portfolio.
- We now can prove the second statement (the opposite direction). In fact, we just reverse the earlier proof.
- Suppose that $\mathbf{X}_z$ be an m-v portfolio, i.e., we can write

$$\mathbf{X}_z = \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_z \end{pmatrix}. \tag{3}$$

- For any pair R_a and R_b, we can find a real number α such that $\alpha R_a + (1-\alpha) R_b = R_z$. Inf act, we set $\alpha = \frac{R_z - R_b}{R_a - R_b}$.
- We can now rewrite the above equation as

$$\begin{aligned} \mathbf{X}_z &= \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ \alpha R_a + (1-\alpha) R_b \end{pmatrix} \\ &= \alpha \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_a \end{pmatrix} \\ &+ (1-\alpha) \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \bar{\mathbf{R}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_b \end{pmatrix} \end{aligned}$$

Two Fund Theorem



Tangency Portfolio with Risk-Free Asset

- With the risk-free asset, the optimal portfolio selection problem becomes

$$\min_{\mathbf{X}} \mathbf{X}^T \Sigma \mathbf{X} \tag{4}$$

subject to

$$\mathbf{X}^T \left(\bar{\mathbf{R}} - R_f \mathbf{1} \right) = R_t - R_f. \tag{5}$$

where R_f is the risk free rate. Note that total weight on risky assets does not need to be equal to 1. If $\mathbf{1}^T \mathbf{X} > 1$, the investor must borrow, and vice versa.

- The Lagrangian now is

$$\mathcal{L} = \mathbf{X}^T \Sigma \mathbf{X} + \mu \left(R_t - R_f - \mathbf{X}^T \left(\bar{\mathbf{R}} - R_f \mathbf{1} \right) \right) \tag{6}$$

Tangency Portfolio with Risk-Free Asset

- The optimal condition or FOC with respect to $\mathbf{X}$ is

$$2\Sigma\mathbf{X} - \mu\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right) = 0 \Rightarrow \mathbf{X} = \frac{\mu}{2}\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right) \tag{7}$$

- Similarly to the above problem we first solve for μ:

$$R_t - R_f = \mathbf{X}^T\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right) = \frac{\mu}{2}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)^T\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)$$

  Thus,

$$\frac{\mu}{2} = \frac{R_t - R_f}{\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)^T\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)} \tag{8}$$

- Thus, we get

$$\mathbf{X} = \frac{(R_t - R_f)\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)}{\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)^T\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)}$$

Tangency Portfolio with Risk-Free Asset

- The tangency portfolio $\mathbf{X}_t$ is such that

$$\mathbf{1}^T\mathbf{X}_t = 1 \tag{9}$$

- Using this condition with the optimal portfolio, we get

$$R_t - R_f = \frac{\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)^T \Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)}{\mathbf{1}^T\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)} \tag{10}$$

- Hence,

$$\mathbf{X}_t = \frac{\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)}{\mathbf{1}^T\Sigma^{-1}\left(\bar{\mathbf{R}} - R_f\mathbf{1}\right)} \tag{11}$$

Efficient Frontier

- Finding an efficient frontier when
  - short sales are allowed and riskless borrowing and lending is possible,
  - short sales are allowed but riskless borrowing and lending is not possible,
  - short sales are not allowed but riskless borrowing and lending is possible,
  - neither short sales nor riskless borrowing and lending are possible.
- Roles of a riskless asset (or riskless lending and borrowing).
- Roles of short sales.

Efficient Frontier with Riskless Lending and Borrowing

- Let riskless be R_F. Let R_A be the return of portfolio of risky assets.
- The average return of the combination between A and riskless asset is

$$\bar{R}_C = (1 - X)R_F + X\bar{R}_A \tag{12}$$

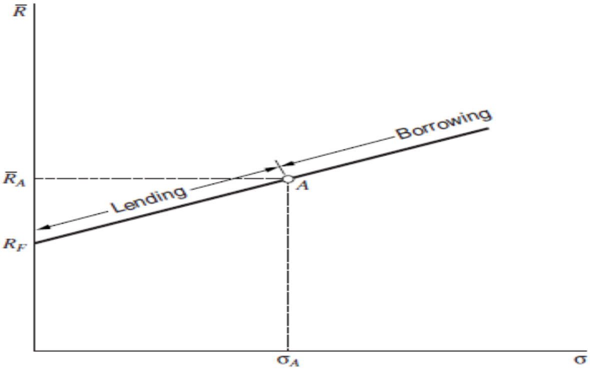
- Variance is

$$\sigma_C = \sqrt{(1-X)^2\sigma_F^2 + X^2\sigma_A^2 + 2X(1-X)\sigma_A\sigma_F\rho_{FA}} = X\sigma_A \tag{13}$$

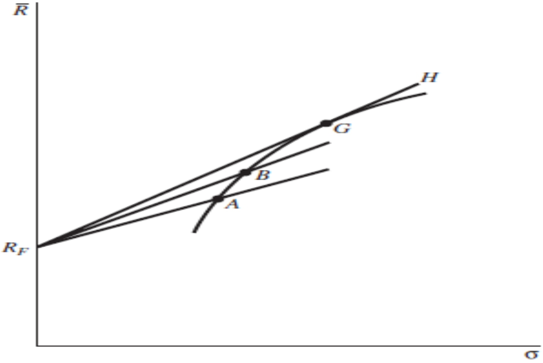
- Hence,

$$\bar{R}_C = R_F + \frac{\bar{R}_A - R_F}{\sigma_A}\sigma_C \tag{14}$$

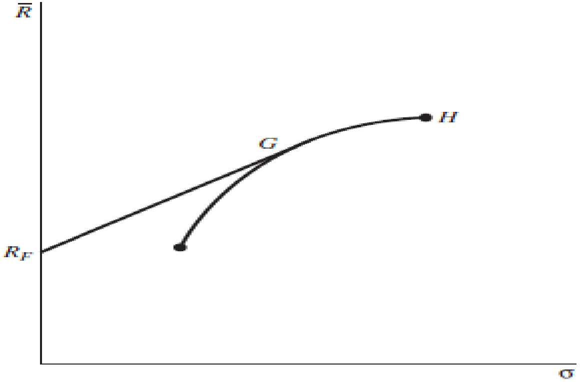
Efficient Frontier with Riskless Lending and Borrowing



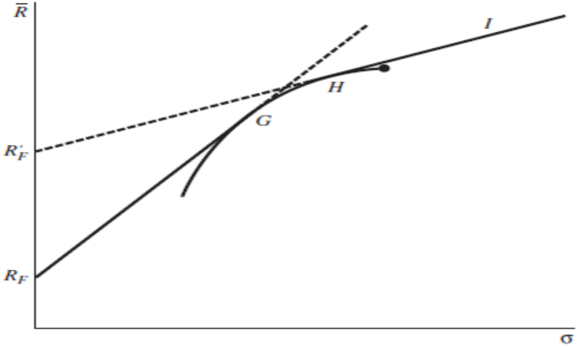
Efficient Frontier with Riskless Lending and Borrowing



Efficient Frontier with Riskless Lending but not Borrowing



Efficient Frontier with Riskless Lending and Borrowing at Different Rates



An Example with Bonds and Stocks

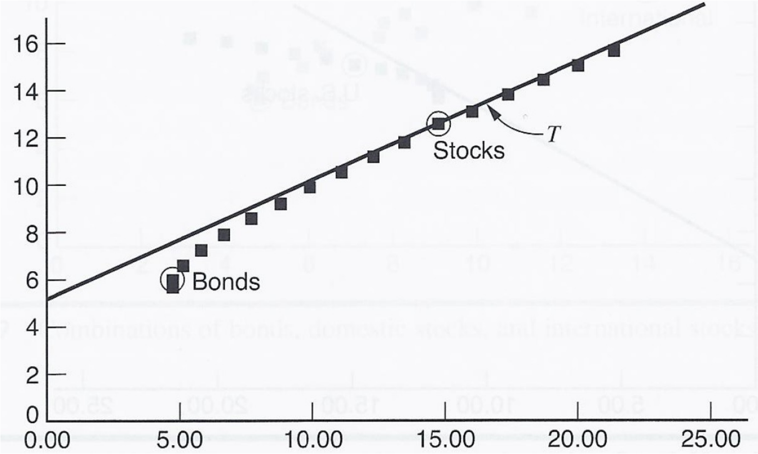
- Returns and variances of bonds and stocks are

$$\bar{R}_S = 12.5\%, \sigma_S = 14.9\% \rho_{S,B} = 0.45, \bar{R}_B = 6\% \sigma_B = 4.8\% \quad (15)$$

- Now assume that the investor and borrow and lend at 5 %. Then we can identify tangent portfolio T. The new efficient frontier is now a straight line.

$$\bar{R}_P = 5 + 0.50\sigma_P \quad (16)$$

Efficient Frontier with Bonds and Stocks

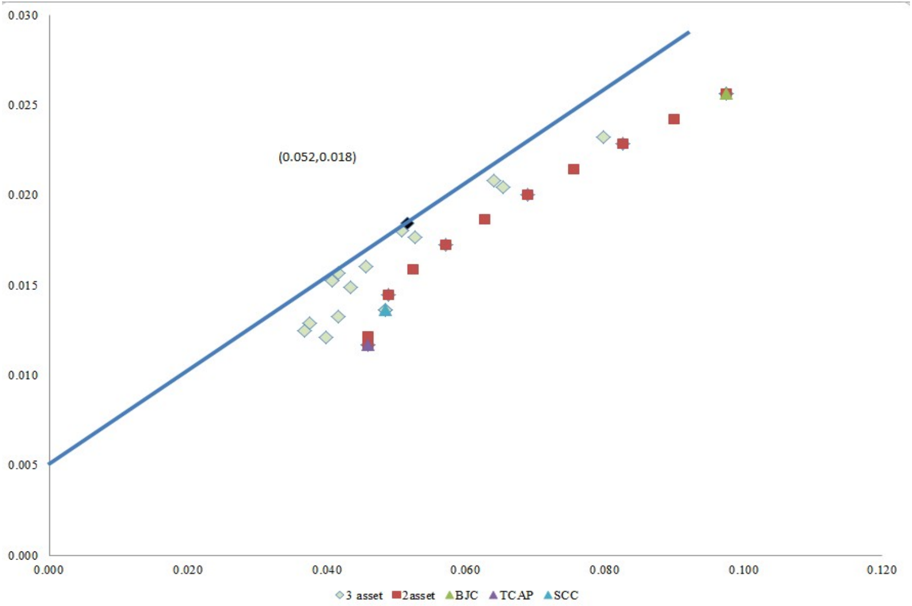


An Example with Bonds and Stocks (Con't)

- From the graph, $\bar{R}_T = 13.54\%$ and $\sigma_T = 16.95\%$
- We can find the corresponding portfolio weight X_S for the tangent portfolio.

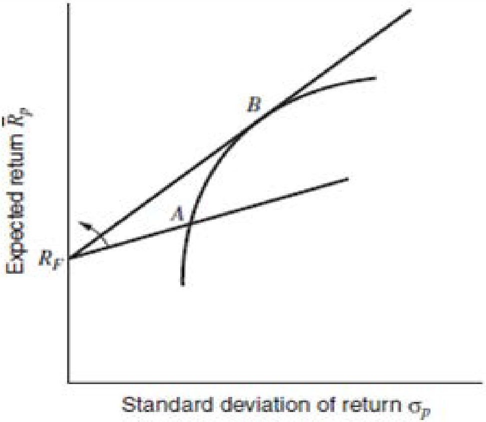
$$13.54 = X_S(2.5) + (1 - X_S)6 \Rightarrow X_S = 116\%, X_B = -16\% \qquad (17)$$

Efficient Frontier with Bonds and Stocks



Short Sales and Riskless implies Maximum Slope

- The existence of riskless asset implies that the efficient frontier in the mean-standard deviation space is the line between the riskless asset and the portfolio of risky assets that gives the **maximum slope** of the line.
- The efficient frontier is the line passing through R_F and B.



Optimal Portfolio Problem with Short Sales and Riskless Asset

- Mathematically, we can find the efficient frontier by solving the following problem.
- The problem is to find a portfolio of risky assets P, whose mean return and standard deviation are $\bar{R}_P$ and σ_P respectively, that maximize the slope

$$\max_{X_i} \frac{\bar{R}_P - R_F}{\sigma_P} \tag{18}$$

subject to

$$\sum_{i=1}^{N} X_i = 1 \tag{19}$$

Mean and Standard Deviation

- The mean return of portfolio P is given by

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i \tag{20}$$

where $\bar{R}_i$ is the mean return of asset i.

- The standard deviation of portfolio P is given by

$$\sigma_P = \left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} X_i X_j \sigma_{ij} \right]^{\frac{1}{2}} \tag{21}$$

Solving for Optimal Portfolio

- The problem can be written as

$$\max_{X_i} \underbrace{\left[\sum_{i=1}^{N} X_i \left(\bar{R}_i - R_F\right)\right]}_{F_1(\mathbf{X})} \underbrace{\left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} X_i X_j \sigma_{ij}\right]^{-\frac{1}{2}}}_{F_2(\mathbf{X})} \qquad (22)$$

- Solving this: using first order conditions (FOCs): differentiating the objective function with respect to a choice variable and take it equal to zero.
- The FOC w.r.t. X_k is

$$\frac{\partial F_1 \times F_2}{\partial X_k} = 0 \Longrightarrow F_1 \frac{\partial F_2}{\partial X_k} + F_2 \frac{\partial F_1}{\partial X_k} = 0 \qquad (23)$$

Solving for Optimal Portfolio: More Details

- Each derivative is

$$\frac{\partial F_1}{\partial X_k} = \bar{R}_k - R_F$$

and

$$\frac{\partial F_2}{\partial X_k} = -\frac{1}{2}\left(\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}^{N} X_i X_j \sigma_{ij}\right)^{-\frac{3}{2}}\left(2X_k\sigma_k^2 + 2\sum_{j\neq k} X_j\sigma_{jk}\right)$$

Solving for Optimal Portfolio: More Details

- Putting these together:

$$
\begin{aligned}
0 &= \left[\bar{R}_P - R_F\right]\left(-\frac{1}{2}\right)\left(\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}^{N} X_i X_j \sigma_{ij}\right)^{-\frac{3}{2}} \\
&\times \left(2X_k\sigma_k^2 + 2\sum_{j\neq k} X_j\sigma_{jk}\right) \\
&+ \left[\bar{R}_k - R_F\right]\left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}^{N} X_i X_j \sigma_{ij}\right]^{-\frac{1}{2}} \\
&= -\left[\bar{R}_P - R_F\right]\sigma_P^{-3}\left(X_k\sigma_k^2 + \sum_{j\neq k} X_j\sigma_{jk}\right) + \left[\bar{R}_k - R_F\right]\sigma_P^{-1}
\end{aligned}
$$

Solving for Optimal Portfolio: More Details

- Multiplying this equation by σ_P and rearranging the terms give

$$0 = -\underbrace{\frac{\bar{R}_P - R_F}{\sigma_P^2}}_{\lambda_P}\left(X_k\sigma_k^2 + \sum_{j\neq k} X_j\sigma_{jk}\right) + \left[\bar{R}_k - R_F\right]$$

which can be rewritten in a compact form as, for each asset k,

$$\bar{R}_k - R_F = \underbrace{\lambda_P X_k}_{Z_k}\sigma_k^2 + \sum_{j\neq k}\underbrace{\lambda_P X_j}_{Z_j}\sigma_{jk}$$

$$\bar{R}_k - R_F = Z_k\sigma_k^2 + \sum_{j\neq k} Z_j\sigma_{jk} \tag{24}$$

System of Simultaneous Equations

- After collecting all FOC for every asset together, we will end up with a system of simultaneous equations:

$$
\begin{aligned}
\bar{R}_1 - R_F &= Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} + \ldots + Z_N\sigma_{1N} \\
\bar{R}_2 - R_F &= Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23} + \ldots + Z_N\sigma_{2N} \\
&\vdots \\
\bar{R}_N - R_F &= Z_1\sigma_{1N} + Z_2\sigma_{12} + Z_3\sigma_{1N} + \ldots + Z_N\sigma_N^2
\end{aligned}
$$

System of Simultaneous Equations

- This system of simultaneous equations can be written in a matrix form as

$$\bar{\mathbf{R}} - R_F \mathbf{1} \;=\; \Sigma \times \mathbf{Z}$$

where

$$\bar{\mathbf{R}} \;=\; \begin{pmatrix} \bar{R}_1 \\ \vdots \\ \bar{R}_N \end{pmatrix},\ \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},\ \mathbf{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_N \end{pmatrix},$$

$$\Sigma \;=\; \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \dots & \sigma_N^2 \end{pmatrix}$$

Recovering the Optimal Portfolio

- In principle, we will be able to solve for Z_i using several methods, e.g.,
  1. inverse matrix
  $$\mathbf{Z} = \Sigma^{-1}\left(\bar{\mathbf{R}} - R_F\mathbf{1}\right) \tag{25}$$
  2. repetitive substitution (see example)
- The solution of this mathematical problem is Z_i but what we really want is X_i. How can we get X_i?
- From $Z_i = \lambda_P X_i$, we can show that
$$\sum_i Z_i = \lambda_P \sum_i X_i = \lambda_P \tag{26}$$
- Hence we can recover the optimal portfolio $\mathbf{X}$ from $\mathbf{Z}$ using
$$X_k = \frac{Z_k}{\lambda_P} = \frac{Z_k}{\sum_i Z_i} \tag{27}$$

Example

- Suppose there are three risky assets, say CP (asset 1), Centrals (asset 2), and PTT (asset 3).
- Using past information, we can calculate mean returns and variance covariance matrix of these assets as

$$
\begin{aligned}
\bar{\mathbf{R}} &= \begin{pmatrix} 14 \\ 8 \\ 20 \end{pmatrix}, \\
\Sigma &= \begin{pmatrix} 6 \times 6 & 0.5 \times 6 \times 3 & 0.2 \times 6 \times 15 \\ 0.5 \times 6 \times 3 & 3 \times 3 & 0.4 \times 3 \times 15 \\ 0.2 \times 6 \times 15 & 0.4 \times 3 \times 15 & 15 \times 15 \end{pmatrix}
\end{aligned}
$$

- Suppose that the riskless rate is 5 %.

Example

- Hence, a system of equations for this problem is

$$\begin{aligned}
9 &= 36Z_1 + 9Z_2 + 18Z_3 \\
3 &= 9Z_1 + 9Z_2 + 18Z_3 \\
15 &= 18Z_1 + 18Z_2 + 225Z_3
\end{aligned}$$

- The solution is

$$\mathbf{Z} = \begin{pmatrix} \frac{14}{63} \\ \frac{1}{63} \\ \frac{3}{63} \end{pmatrix}$$

Example

- We can then find portfolio weight $\mathbf{X}$ as

$$X_1 = \frac{14}{18},\ X_2 = \frac{1}{18},\ X_3 = \frac{3}{18}$$

- The mean return of the optimal portfolio is

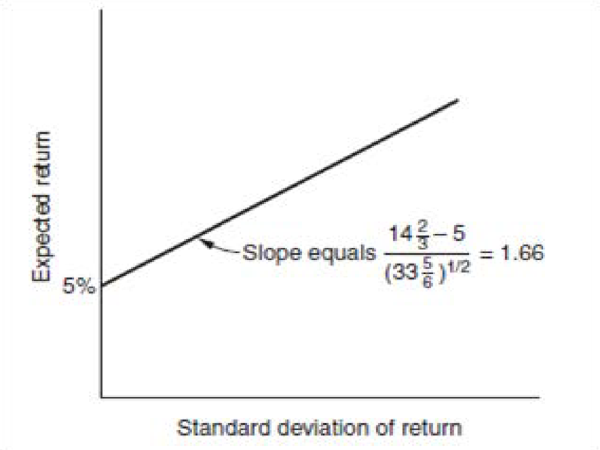
$$\bar{R}_P = \frac{14}{18} \times 14 + \frac{1}{18} \times 8 + \frac{3}{18} \times 20 = 14.67\%$$

- The variance of the optimal portfolio is

$$\sigma_P^2 = \mathbf{X}^T \Sigma \mathbf{X} = 33.83\%$$

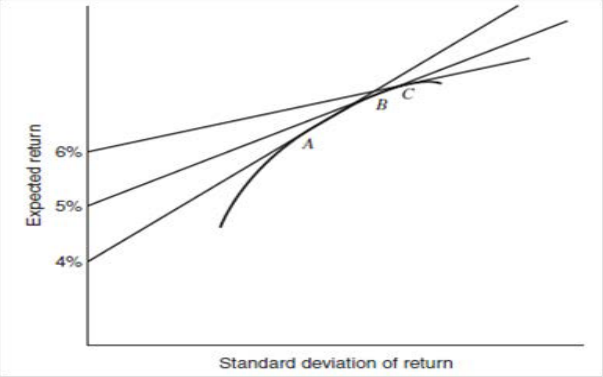
Example: Solution

- The slope of the efficient frontier is equal to 1.66.



Short Sales without Riskless

- We will now consider a case where short sales are allowed but there is no riskless asset.
- We can use the same technique as before but with an assumed rate $\tilde{R}_F$. Different assumed rates will lead to different efficient portfolios.



Example Continued.

- Suppose that the riskless rate is now $\tilde{R}_F = 2$.
- The system of equations now becomes

$$\begin{aligned} 12 &= 36Z_1 + 9Z_2 + 18Z_3 \\ 6 &= 9Z_1 + 9Z_2 + 18Z_3 \\ 18 &= 18Z_1 + 18Z_2 + 225Z_3 \end{aligned}$$

whose solution is

$$Z_1 = \frac{42}{189}, Z_2 = \frac{72}{189}, Z_3 = \frac{6}{189} \text{ and } X_1 = \frac{7}{20}, X_2 = \frac{12}{20}, X_3 = \frac{1}{20}$$

and

$$\bar{R}_P = 10.7, \sigma_P^2 = 13.70$$

Example Continued

- In principle, we need to find only two of them (Two Fund Theorem).
- That is, any combination of these two portfolios (which themselves are assets) is on the efficient frontier.
- For example: put $50 - 50$ weight. We can show that $\sigma_P^2 = 21.859$.
- Then we can find the covariance between the two portfolios: using

$$\sigma_P^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}$$

  This leads to $\sigma_{12} = 19.95$.
- With the information of expected returns, variances, and covariance between the two portfolios, we can trace out the whole frontier.

Riskless but No Short Sales

- We will now consider a case where short sales are not allowed but there is a riskless asset.
- In principle, an efficient portfolio problem is a constrained maximization problem. In this case, we can write

$$\max_{X_i} \frac{\bar{R}_P - R_F}{\sigma_P} \tag{28}$$

subject to

$$\sum_i X_i = 1 \tag{29}$$

$$X_i \geq 0, \forall i \tag{30}$$

where the last one represents the no short-sales constraint.

Riskless but No Short Sales: Example

- Consider again the example with three assets and risk-free rate $R_F = 5\%$.
- Recall that the efficient portfolio in this case is

$$X_1 = \frac{14}{18}, \ X_2 = \frac{1}{18}, \ X_3 = \frac{3}{18}$$

  Remember that this solution is solved under an assumption that short sales are allowed.
- What if we now impose the no short-sales constraint, should we get a different answer?

Example II

- Suppose there are three risky assets, say CP (asset 1), Centrals (asset 2), and PTT (asset 3).
- Using past information, we can calculate mean returns and variance covariance matrix of these assets as

$$
\begin{aligned}
\bar{\mathbf{R}} &= \begin{pmatrix} 14 \\ 8 \\ 20 \end{pmatrix}, \\
\Sigma &= \begin{pmatrix} 6 \times 6 & 0.5 \times 6 \times 10 & 0.2 \times 6 \times 15 \\ 0.5 \times 6 \times 10 & 10 \times 10 & 0.4 \times 10 \times 15 \\ 0.2 \times 6 \times 15 & 0.4 \times 10 \times 15 & 15 \times 15 \end{pmatrix}
\end{aligned}
$$

- Suppose that the riskless rate is 5 %.

Example

- Hence, a system of equations for this problem is

$$\begin{aligned} 9 &= 36Z_1 + 30Z_2 + 18Z_3 \\ 3 &= 30Z_1 + 100Z_2 + 60Z_3 \\ 15 &= 18Z_1 + 60Z_2 + 225Z_3 \end{aligned}$$

- The solution is

$$\mathbf{Z} = \begin{pmatrix} 0.3000 \\ -0.1019 \\ 0.0698 \end{pmatrix}$$

Example

- We can then find portfolio weight **X** as

$$X_1 = 1.1197,\ X_2 = -0.3803,\ X_3 = 0.2607$$

- But we do not allow for a negative weight! What should we do?

More General Efficient Portfolio Problem

- This problem started from the seminal work by Markowitz (1959).

$$\min_{\mathbf{X}} \sum_i X_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} X_i X_j \sigma_{ij} \tag{31}$$

subject to

$$\begin{aligned} \sum_i X_i &= 1 && (32) \\ \sum_i X_i \bar{R}_i &\geq \bar{R}_P && (33) \\ X_i &\geq 0, \forall i && (34) \\ \sum_i X_i d_i &\geq D && (35) \end{aligned}$$

where the last constraint is the so called dividend requirement constraint.

- The role of a riskless asset is to simplify the objective function as a slope.

CAPM: A first look

- From the tangency portfolio, we have

$$\sigma_t^2 = \mathbf{X}^T \Sigma \mathbf{X} = \frac{R_t - R_f}{\mathbf{1}^T \Sigma^{-1} \left(\bar{\mathbf{R}} - R_f \mathbf{1} \right)} \tag{36}$$

  Hence we can write

$$\mathbf{1}^T \Sigma^{-1} \left(\bar{\mathbf{R}} - R_f \mathbf{1} \right) = \frac{R_t - R_f}{\sigma_t^2} \tag{37}$$

- Similarly, covariance between the tangency portfolio and all assets is given by

$$Cov_t = \Sigma \mathbf{X} = \frac{\bar{\mathbf{R}} - R_f \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \left(\bar{\mathbf{R}} - R_f \mathbf{1} \right)} \tag{38}$$

CAPM: A first look

- Combining the last two equations lead to CAPM

$$\bar{\mathbf{R}} - R_f \mathbf{1} = \frac{Cov_t}{\sigma_t^2} (R_t - R_f) \tag{39}$$

- We now set the ratio between covariance and variance as β, e.g., $\beta_i = \frac{Cov(R_1, R_t)}{\sigma_t^2}$, where R_t is a return process of the tangency portfolio (a random variable).
- We now have a CAPM with the tangency portfolio as a market portfolio:

$$\bar{\mathbf{R}} - R_f \mathbf{1} = \beta_t (R_t - R_f) \tag{40}$$

where $\beta_t = (\beta_i)_{i=1}^n$.