Single-Index Model

Main Issues

- Implementation of portfolio theory: this process requires correlation structure of security returns.

Correlation Structure of Security Returns is a Key Input

- Recall from Mean-Variance Portfolio:

$$\mathbf{Z} = \Sigma^{-1}\left(\bar{\mathbf{R}} - R_F \mathbf{1}\right) \tag{1}$$

  This implies that correlation structure of security returns, Σ^{-1}, is a key input to an optimal portfolio problem.

- Consider a portfolio P. Its expected return and expected variance (or risk) are

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i \tag{2}$$

$$\sigma_p = \left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq \mathrm{i}}}^{N} X_i X_j \sigma_i \sigma_j \rho_{ij}\right]^{\frac{1}{2}} \tag{3}$$

Correlation Structure of Security Returns is a Key Input

- How many correlations do we need to estimate when there are N assets? Answer is $\frac{N(N-1)}{2}$.
- In order to estimate correlation of each pair of assets, ρ_{ij}, we need to have data of historical returns of both assets.
- A classical problem is that a finance firm traditionally organize their analysts along industry line. That is, each analyst will know very well about one asset only.
- Therefore, calculating correlation requires coordination effort (which is costly as well).
- In my opinion, this may not be an issue with the database technology now a day?
- But still this model may bring more predictive power than simple covariance calculation!
- Anyway, suppose it is still a problem for now!

Market Return is the Single Index

- To solve this coordination requirement problem, we need to find a model that all correlations can be calculated or derived from simple statistics that each analyst can provide without coordinating with each other.
- One solution is: to assume that return of each asset i, R_i depends on the market return, R_m:

$$R_i = a_i + \beta_i R_m \tag{4}$$

where

a_i is the component of security i's performance — a random variable (with $a_i = \alpha_i + \epsilon_i$).

ϵ_i is a random error— a random variable.

R_m is the rate of return on the market index — a random variable.

β_i is a constant that measures the expected change in R_i given a change in R_m (sensitivity).

Regression Equation

- The equation for the return on a security can be written as

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i \tag{5}$$

This is a linear regression equation.

- Key Assumptions:
  1. Error term ϵ_i is unrelated/uncorrelated to the market return R_m:

$$E[\epsilon_i(R_m - E(R_m))] = 0, \text{ for all stocks } i, \tag{6}$$

  2. Securities are related through common response to market only:

$$E(\epsilon_i \epsilon_j) = 0, \text{ for all pairs of stocks } i \neq j, \tag{7}$$

- By construction:
  1. Mean of ϵ_i is zero:

$$E(\epsilon_i) = 0, \text{ for all stocks } i. \tag{8}$$

Regression Equation

- By definition:
  1. Variance of ϵ_i is

$$E(\epsilon_i)^2 = \sigma^2 = \sigma^2_{\epsilon_i}, \text{ for all stocks } i, \tag{9}$$

  2. Variance of R_m is

$$E\left(R_m - E\left(R_m\right)\right)^2 = \sigma^2_m, \text{ for all stocks } i, \tag{10}$$

Covariance Structure from the Single Index

- The mean of a security i return is

$$E(R_i) = E(\alpha_i + \beta_i R_m + \epsilon_i) = \alpha_i + \beta_i E(R_m) \tag{11}$$

- The Variance of a security i return is

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\epsilon_i}^2 \tag{12}$$

- The covariance of returns between securities i and j,

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2. \tag{13}$$

Example

Table: Decomposition of Returns for the Single-Index Model

Month	Return on Stock	Return on Market
1	10	4
2	3	2
3	15	8
4	9	6
5	3	0

- Suppose $\beta_i = 1.5$, what are α_i and $\sigma^2_{\epsilon i}$ (the size of the error)?
- $\alpha_i = 2$ and $\sigma^2_{\epsilon i} = 2.8$.

Covariance Structure of a Portfolio

- Let $(X_i)_{i=1}^{N}$ be a portfolio P. Its expected return is

$$E(R_p) = \sum_{i=1}^{N} X_i E(R_i) \tag{14}$$

  Using the result above, we have

$$E(R_p) = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i E(R_m) \tag{15}$$

- Its variance is

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \sigma_{ij} \tag{16}$$

  Using the result above, we have

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\epsilon i}^2 \tag{17}$$

Characteristics of the Single-Index Model

- Using the result in the previous slide, we can show that

$$\beta_p = \sum_{i=1}^{N} X_i \beta_i \tag{18}$$

and

$$\alpha_p = \sum_{i=1}^{N} X_i \alpha_i \tag{19}$$

Hence, we can write

$$E(R_p) = \alpha_p + \beta_p E(R_m) \tag{20}$$

If P is the market portfolio, then its $\beta_m = 1$ and $\alpha_m = 0$.

Diversifiable Risk

- The risk of an individual portfolio is

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\epsilon i}^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq \mathrm{i}}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 \quad (21)$$

which can be further rearrange as

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \frac{1}{N} \left(\sum_{i=1}^{N} \frac{1}{N} \sigma_{\epsilon i}^2 \right) \quad (22)$$

- The average residual risk term is going to be very small when the number of securities N is sufficiently large. Hence, it is common to refer to $\sigma_{\epsilon i}^2$ as diversifiable or nonsystematic risk.

Systematic Risk

- On the other hand, if we assume that the diversifiable term is zero, then portfolio P's risk is

$$\sigma_p^2 = \sigma_m^2 \left[\sum_{i=1}^{N} X_i \beta_i \right]^2 \qquad (23)$$

That is, β_i measures security i's systematic or nondiversifiable risk.

Estimating Historical βs

- Recall:

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i \tag{24}$$

  This is a linear regression equation.

- We can use historical or past time-series data on security R_{it} and market return R_{mt} to estimate β_i:

$$\beta_i = \frac{Cov\left(R_i, R_m\right)}{\sigma_m^2} = \frac{\sum_{t=1}^{60}\left[(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})\right]}{\sum_{t=1}^{60}\left(R_{mt} - \bar{R}_{mt}\right)^2} \tag{25}$$

$$\alpha_i = E\left(R_i\right) - \beta_i E\left(R_m\right) = \bar{R}_i - \beta_i \bar{R}_m \tag{26}$$

- Of course, in practice, this is a very easy task now a day. You can just run it on a statistical program such as STATA.

Accuracy of Historical βs

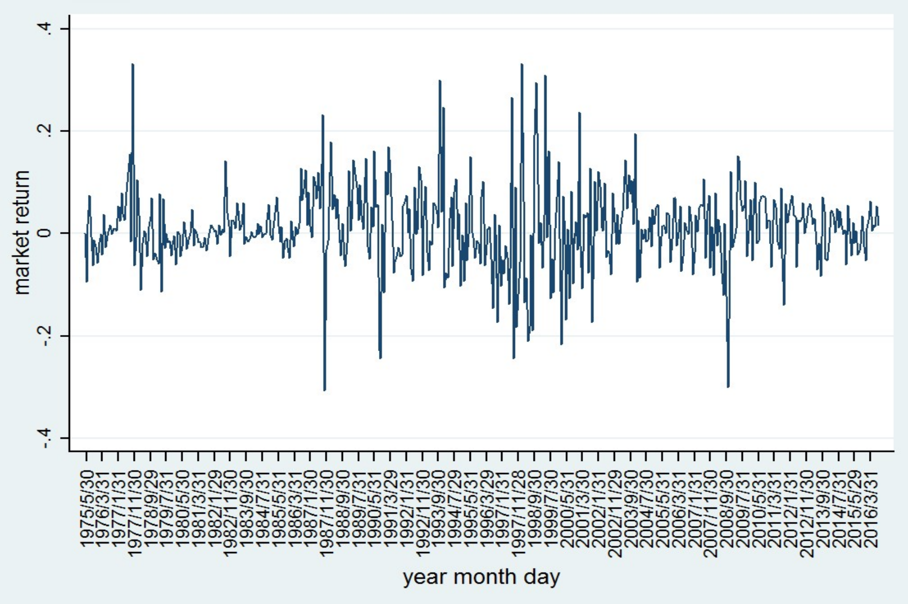
- If the β_i is accurate, we should see a high correlation between β_i from different periods. See the Table below (from Blume (1970))

Table: Association of Betas Over Time

Number of Securities in the Portfolio	Correlation Coefficient	Coefficient of Determination (R^2)
1	0.60	0.36
2	0.73	0.53
20	0.97	0.95
35	0.97	0.95
50	0.98	0.96

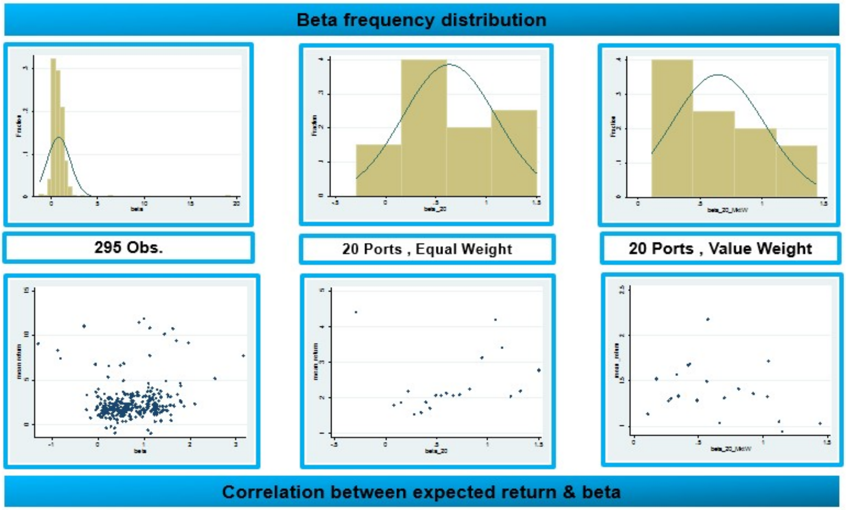
- β on individual securities are not so accurate (corr is about 0.60).
- β of a large portfolio is very accurate.
- What should we do if we really need a β of an individual security?

βs in Thailand

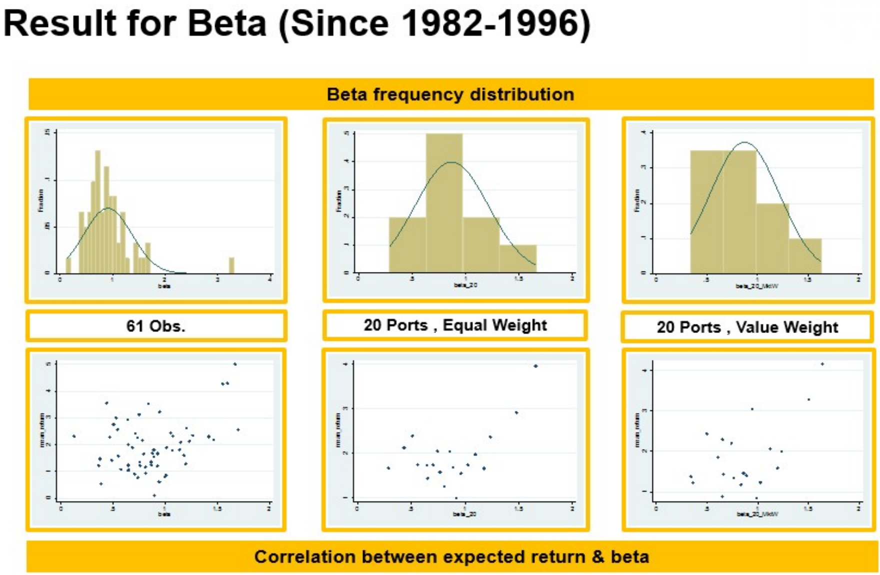


βs in Thailand

Result for Beta (Since 2001-2015)

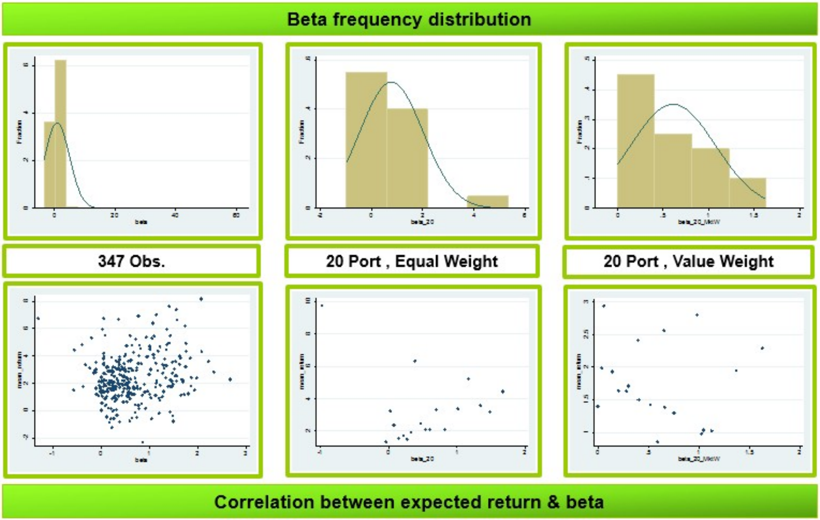


βs in Thailand



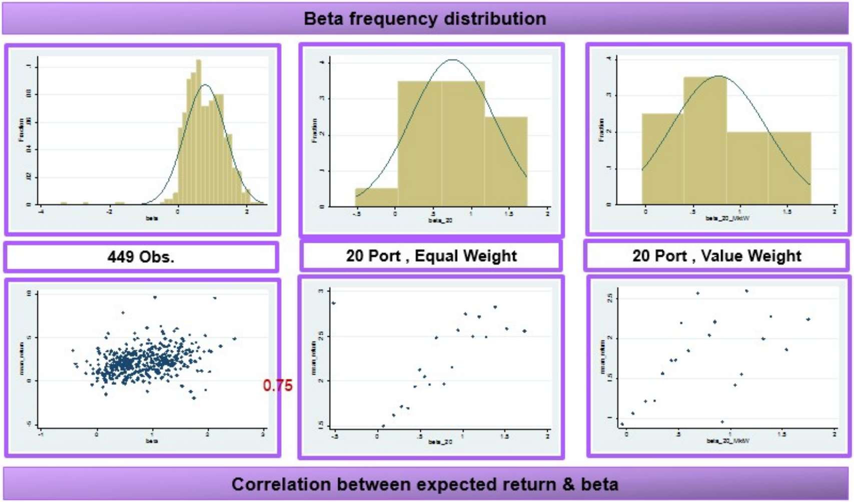
βs in Thailand

Result for Beta (Since 2001-2006)



βs in Thailand

Result for Beta (Since 2009-2015)



Adjusting Historical βs

- We will need to account for the fact that β last period may have a limit power to predict β this period.
- Blume's Technique: this approach is based on a simple regression, e.g.,

$$\beta_{i2} = 0.343 + 0.677\beta_{i1} \tag{27}$$

where β_{i1} stands for the β of stock/asset i for the earlier period, and β_{i2} stands for the β of stock/asset i for the later period. This is a result of a (cross-sectional) regression.

Adjusting Historical βs: Vasecek's Technique

- Vasecek's Technique: this approach is based on a Bayesian estimation technique (use information on variances), i.e.,

$$\beta_{i2} = \frac{\sigma^2_{\hat{\beta}_{i1}}}{\sigma^2_{\bar{\beta}_1} + \sigma^2_{\hat{\beta}_{i1}}} \bar{\beta}_1 + \frac{\sigma^2_{\bar{\beta}1}}{\sigma^2_{\bar{\beta}_1} + \sigma^2_{\hat{\beta}_{i1}}} \hat{\beta}_{i1} \tag{28}$$

where

$\bar{\beta}_1$ stands for a weighted average of β over all securities using historical data,

$\sigma^2_{\bar{\beta}1}$ stands for a **cross-sectional variance** of $\hat{\beta}_{i1}$ across all securities using historical data, and

$\sigma^2_{\hat{\beta}_{i1}}$ stands for the square of the **standard error** of the estimate of β for a security i (variance of an estimator).

Note: Standard Error of a Regression Estimator

- Standard error of the estimator $\hat{\beta}_{i1}$, for population is

$$\sigma^2_{\hat{\beta}_{i1}} = \frac{\sum_{t=1}^{n} \epsilon_{it}^2}{N \sum_{t=1}^{N} (R_{mt} - \bar{R}_m)^2} \tag{29}$$

- Standard error of the estimator $\hat{\beta}_{i1}$, for finite sample N is

$$\sigma^2_{\hat{\beta}_{i1}} = \frac{\sum_{t=1}^{n} \epsilon_{it}^2}{(N - K) \sum_{t=1}^{N} (R_{mt} - \bar{R}_m)^2} \tag{30}$$

where K is the number of parameters of the model (α, β) or $N - K$ is the degree of freedom of the model.

Accuracy of Adjusted βs

- One way to see which approach work better is to compare the predicted correlation of each approach.
- Recall:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j} \tag{31}$$

- Klemkosky and Martin (1975) shows that both adjustment techniques work better than the basic non-adjusted approach.
- In addition, Elton, Gruber, and Urich (1978) show that Bayesian technique predicts the correlation structure the best.

Fundamental βs

- Another way to predict β is to use fundamentals as independent variables or predicting variables.

$$\beta_i = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_N X_N + \epsilon_i \tag{32}$$

where each X_i is one of the N variables hypothesized as affecting Beta.

- Popular fundamentals are
  1. dividend,
  2. asset growth,
  3. leverage (senior securities divided by total assets),
  4. liquidity (current assets divided current liabilities),
  5. asset size,
  6. earning variability.