Single-Index Model

©Kilenthong 2017
Main Issues

- Implementation of portfolio theory: this process requires correlation structure of security returns.
Correlation Structure of Security Returns is a Key Input

- Recall from Mean-Variance Portfolio:
  \[ Z = \Sigma^{-1} (\bar{R} - R_F 1) \] (1)

  This implies that correlation structure of security returns, \( \Sigma^{-1} \), is a key input to an optimal portfolio problem.

- Consider a portfolio \( P \). Its expected return and expected variance (or risk) are

  \[ \bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i \] (2)

  \[ \sigma_p = \left[ \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_i \sigma_j \rho_{ij} \right]^{\frac{1}{2}} \] (3)
How many correlations do we need to estimate when there are \( N \) assets? Answer is \( \frac{N(N-1)}{2} \).

In order to estimate correlation of each pair of assets, \( \rho_{ij} \), we need to have data of historical returns of both assets.

A classical problem is that a finance firm traditionally organize their analysts along industry line. That is, each analyst will know very well about one asset only.

Therefore, calculating correlation requires coordination effort (which is costly as well).

In my opinion, this may not be an issue with the database technology now a day?

But still this model may bring more predictive power than simple covariance calculation!

Anyway, suppose it is still a problem for now!
Market Return is the Single Index

- To solve this coordination requirement problem, we need to find a model that all correlations can be calculated or derived from simple statistics that each analyst can provide without coordinating with each other.

- One solution is: to assume that return of each asset $i$, $R_i$ depends on the market return, $R_m$:

$$R_i = a_i + \beta_i R_m$$  \hspace{1cm} (4)

where

- $a_i$ is the component of security $i$’s performance — a random variable (with $a_i = \alpha_i + \epsilon_i$).
- $\epsilon_i$ is a random error— a random variable.
- $R_m$ is the rate of return on the market index — a random variable.
- $\beta_i$ is a constant that measures the expected change in $R_i$ given a change in $R_m$ (sensitivity).
Regression Equation

The equation for the return on a security can be written as

\[ R_i = \alpha_i + \beta_i R_m + \epsilon_i \]  \hspace{1cm} (5)

This is a linear **regression** equation.

**Key Assumptions:**

1. Error term \( \epsilon_i \) is unrelated/uncorrelated to the market return \( R_m \):

\[ E[\epsilon_i (R_m - E(R_m))] = 0, \text{ for all stocks } i, \]  \hspace{1cm} (6)

2. Securities are related through common response to market only:

\[ E(\epsilon_i \epsilon_j) = 0, \text{ for all pairs of stocks } i \neq j, \]  \hspace{1cm} (7)

By construction:

1. Mean of \( \epsilon_i \) is zero:

\[ E(\epsilon_i) = 0, \text{ for all stocks } i. \]  \hspace{1cm} (8)
Regression Equation

- By definition:
  1. Variance of $\epsilon_i$ is
     \[ E(\epsilon_i)^2 = \sigma^2 = \sigma_{\epsilon_i}^2, \quad \text{for all stocks } i, \]  
    (9)
  2. Variance of $R_m$ is
     \[ E(R_m - E(R_m))^2 = \sigma_m^2, \quad \text{for all stocks } i, \]  
    (10)
The mean of a security $i$ return is

$$ E(R_i) = E(\alpha_i + \beta_i R_m + \epsilon_i) = \alpha_i + \beta_i E(R_m) $$

(11)

The Variance of a security $i$ return is

$$ \sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\epsilon_i}^2 $$

(12)

The covariance of returns between securities $i$ and $j$,

$$ \sigma_{ij} = \beta_i \beta_j \sigma_m^2 $$

(13)
Example

<table>
<thead>
<tr>
<th>Month</th>
<th>Return on Stock</th>
<th>Return on Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table: Decomposition of Returns for the Single-Index Model

- Suppose $\beta_i = 1.5$, what are $\alpha_i$ and $\sigma^2_{\epsilon_i}$ (the size of the error)?
- $\alpha_i = 2$ and $\sigma^2_{\epsilon_i} = 2.8$. 

---

Suppose $\beta_i = 1.5$, what are $\alpha_i$ and $\sigma^2_{\epsilon_i}$ (the size of the error)?

- $\alpha_i = 2$ and $\sigma^2_{\epsilon_i} = 2.8$. 

---

**Example**

**Table**: Decomposition of Returns for the Single-Index Model

<table>
<thead>
<tr>
<th>Month</th>
<th>Return on Stock</th>
<th>Return on Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>
Covariance Structure of a Portfolio

- Let \((X_i)_{i=1}^{N}\) be a portfolio \(P\). Its expected return is

\[
E(R_p) = \sum_{i=1}^{N} X_i E(R_i) \tag{14}
\]

Using the result above, we have

\[
E(R_p) = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i E(R_m) \tag{15}
\]

- Its variance is

\[
\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij} \tag{16}
\]

Using the result above, we have

\[
\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\epsilon i}^2 \tag{17}
\]
Using the result in the previous slide, we can show that

\[ \beta_p = \sum_{i=1}^{N} X_i \beta_i \]  

and

\[ \alpha_p = \sum_{i=1}^{N} X_i \alpha_i \]  

Hence, we can write

\[ E(R_p) = \alpha_p + \beta_p E(R_m) \]  

If \( P \) is the market portfolio, then its \( \beta_m = 1 \) and \( \alpha_m = 0 \).
Diversifiable Risk

• The risk of an individual portfolio is

\[
\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\epsilon_i}^2 + \sum_{i=1}^{N} \sum_{j=1 \atop j \neq i}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 \tag{21}
\]

which can be further rearrange as

\[
\sigma_p^2 = \beta_p^2 \sigma_m^2 + \frac{1}{N} \left( \sum_{i=1}^{N} \frac{1}{N} \sigma_{\epsilon_i}^2 \right) \tag{22}
\]

• The average residual risk term is going to be very small when the number of securities \( N \) is sufficiently large. Hence, it is common to refer to \( \sigma_{\epsilon_i}^2 \) as diversifiable or nonsystematic risk.
On the other hand, if we assume that the diversifiable term is zero, then portfolio $P$’s risk is

$$\sigma_p^2 = \sigma_m^2 \left[ \sum_{i=1}^{N} X_i \beta_i \right]^2$$

(23)

That is, $\beta_i$ measures security $i$’s systematic or nondiversifiable risk.
Estimating Historical $\beta$s

- Recall:

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i$$  \hspace{1cm} (24)

This is a linear regression equation.

- We can use historical or past time-series data on security $R_{it}$ and market return $R_{mt}$ to estimate $\beta_i$:

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\sigma_m^2} = \sum_{t=1}^{60} \frac{[(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})]}{\sum_{t=1}^{60} (R_{mt} - \bar{R}_{mt})^2}$$  \hspace{1cm} (25)

$$\alpha_i = E(R_i) - \beta_i E(R_m) = \bar{R}_i - \beta_i \bar{R}_m$$  \hspace{1cm} (26)

- Of course, in practice, this is a very easy task now a day. You can just run it on a statistical program such as STATA.
Accuracy of Historical $\beta$s

- If the $\beta_i$ is accurate, we should see a high correlation between $\beta_i$ from different periods. See the Table below (from Blume (1970))

<table>
<thead>
<tr>
<th>Number of Securities in the Portfolio</th>
<th>Correlation Coefficient</th>
<th>Coefficient of Determination ($R^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.60</td>
<td>0.36</td>
</tr>
<tr>
<td>2</td>
<td>0.73</td>
<td>0.53</td>
</tr>
<tr>
<td>20</td>
<td>0.97</td>
<td>0.95</td>
</tr>
<tr>
<td>35</td>
<td>0.97</td>
<td>0.95</td>
</tr>
<tr>
<td>50</td>
<td>0.98</td>
<td>0.96</td>
</tr>
</tbody>
</table>

- $\beta$ on individual securities are not so accurate (corr is about 0.60).
- $\beta$ of a large portfolio is very accurate.
- What should we do if we really need a $\beta$ of an individual security?
$\beta$s in Thailand
βs in Thailand

Result for Beta (Since 2001-2015)

Beta frequency distribution

295 Obs.

20 Ports, Equal Weight

20 Ports, Value Weight

Correlation between expected return & beta

Single-Index Model
Result for Beta (Since 1982-1996)

Beta frequency distribution

61 Obs.

20 Ports, Equal Weight

20 Ports, Value Weight

Correlation between expected return & beta

βs in Thailand
βs in Thailand

Result for Beta (Since 2001-2006)

Beta frequency distribution

- 347 Obs.
- 20 Port, Equal Weight
- 20 Port, Value Weight

Correlation between expected return & beta
Result for Beta (Since 2009-2015)

Beta frequency distribution

- 449 Obs.
- 20 Port, Equal Weight
- 20 Port, Value Weight

Correlation between expected return & beta
Adjusting Historical $\beta$s

- We will need to account for the fact that $\beta$ last period may have a limit power to predict $\beta$ this period.

- Blume’s Technique: this approach is based on a simple regression, e.g.,

$$\beta_{i2} = 0.343 + 0.677\beta_{i1} \quad (27)$$

where $\beta_{i1}$ stands for the $\beta$ of stock/asset $i$ for the earlier period, and $\beta_{i2}$ stands for the $\beta$ of stock/asset $i$ for the later period. This is a result of a (cross-sectional) regression.
Vasecek’s Technique: this approach is based on a Bayesian estimation technique (use information on variances), i.e.,

$$
\beta_{i2} = \frac{\sigma^2_{\hat{\beta}_{i1}}}{\sigma^2_{\hat{\beta}_{i1}} + \sigma^2_{\hat{\beta}_{i1}}} \bar{\beta}_1 + \frac{\sigma^2_{\hat{\beta}_{i1}}}{\sigma^2_{\hat{\beta}_{i1}} + \sigma^2_{\hat{\beta}_{i1}}} \hat{\beta}_{i1}
$$

(28)

where

- $\bar{\beta}_1$ stands for a weighted average of $\beta$ over all securities using historical data,
- $\sigma^2_{\hat{\beta}_{i1}}$ stands for a **cross-sectional variance** of $\hat{\beta}_{i1}$ across all securities using historical data, and
- $\sigma^2_{\hat{\beta}_{i1}}$ stands for the square of the **standard error** of the estimate of $\beta$ for a security $i$ (variance of an estimator).
Note: Standard Error of a Regression Estimator

- Standard error of the estimator $\hat{\beta}_{i1}$, for population is

$$\sigma^2_{\hat{\beta}_{i1}} = \frac{\sum_{t=1}^{n} \epsilon_{it}^2}{N \sum_{t=1}^{N} (R_{mt} - \bar{R}_m)^2}$$  \hspace{1cm} (29)

- Standard error of the estimator $\hat{\beta}_{i1}$, for finite sample $N$ is

$$\sigma^2_{\hat{\beta}_{i1}} = \frac{\sum_{t=1}^{n} \epsilon_{it}^2}{(N - K) \sum_{t=1}^{N} (R_{mt} - \bar{R}_m)^2}$$  \hspace{1cm} (30)

where $K$ is the number of parameters of the model $(\alpha, \beta)$ or $N - K$ is the degree of freedom of the model.
Accuracy of Adjusted $\beta$s

- One way to see which approach work better is to compare the predicted correlation of each approach.
- Recall:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j}$$  (31)

- Klemkosky and Martin (1975) shows that both adjustment techniques work better than the basic non-adjusted approach.
- In addition, Elton, Gruber, and Urich (1978) show that Bayesian technique predicts the correlation structure the best.
Fundamental $\beta$s

- Another way to predict $\beta$ is to use fundamentals as independent variables or predicting variables.

$$\beta_i = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_N X_N + \epsilon_i$$  \hspace{1cm} (32)

where each $X_i$ is one of the $N$ variables hypothesized as affecting Beta.

- Popular fundamentals are
  1. dividend,
  2. asset growth,
  3. leverage (senior securities divided by total assets),
  4. liquidity (current assets divided current liabilities),
  5. asset size,
  6. earning variability.