The Capital Asset Pricing Model (CAPM)

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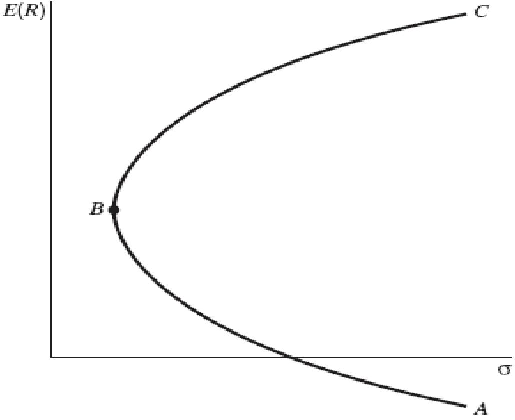
Main Issues

- What is an equilibrium implication if all investors construct portfolios as we studied?
- How should we measure risk of any asset?

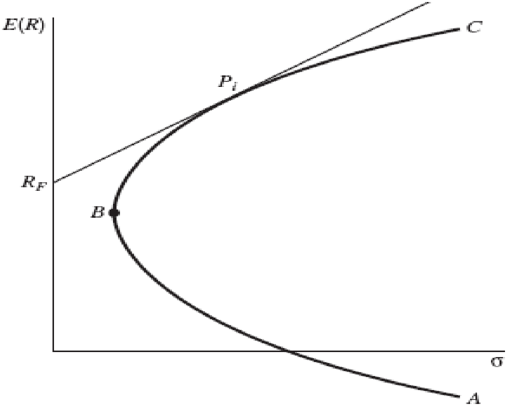
CAPM: Simple Derivation

- The efficient frontier without riskless asset.



CAPM: Simple Derivation

- The efficient frontier with riskless asset. The straight line is called the capital market line.



CAPM: Simple Derivation

- The equation of the capital market line is

$$\bar{R}_e = R_F + \frac{\bar{R}_m - R_F}{\sigma_m}\sigma_e \tag{1}$$

where σ_e is an efficient portfolio on the capital market line.

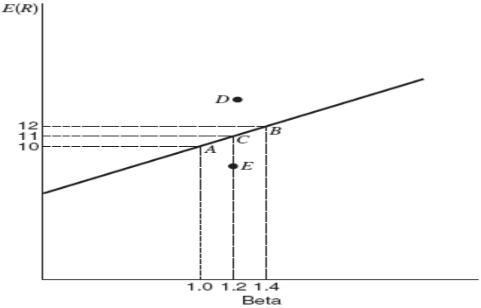
- $\frac{\bar{R}_m - R_F}{\sigma_m}$ represents the market price of risk.
- σ_e represents the amount of risk.
- That is,

$$\text{Expected return} = (\text{Market price of risk}) \times (\text{Amount of risk}) \tag{2}$$

- Problem: This equation does not describe equilibrium return on nonefficient portfolios or individual securities.

Only $\bar{R}_i$ and β_i Matter

- From Single-Index Model, the investor should hold a very well-diversified portfolio. Therefore, only relevant risk is systematic risk measured by β.
- The only dimensions of a security that need be of concern are expected return $\bar{R}_i$ and β_i.



Nonefficient Portfolios and Arbitrage Opportunity

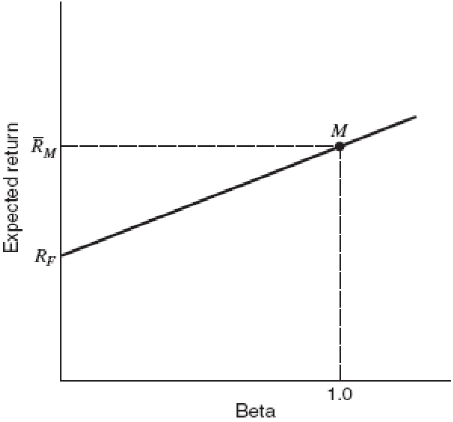
- From the above figure, one can imagine that they can make money from the following arbitrage strategy:

	Cash Invested	Expected Return	β
Portfolio C	-100	-11	-1.2
Portfolio D	+100	+13	+1.2
Arbitrage Portfolio	0	2	0

- Note: we use $\beta_P = \sum_i X_i \beta_i$.
- Main point: there is a portfolio involving zero risk and zero net investment that has a positive expected return. There is an arbitrage opportunity. This should not be the case in an equilibrium.

Nonefficient Portfolios and Arbitrage Opportunity

- Therefore, all investors must hold efficient portfolios.



CAPM Equation

- The security market line can be represented by

$$\bar{R}_i = R_F + \left(\bar{R}_m - R_F\right)\beta_i \tag{3}$$

- Key Insight: systematic risk (measured by β) is the only important ingredient in determining expected returns and that nonsystematic risk plays no role.
- In other words, investors get rewarded for bearing systematic risk.
- Important: this implication is empirically testable.

CAPM Equation: Alternative

- We can represent risk by covariance of asset return and market return:

$$\bar{R}_i = R_F + \left(\frac{\bar{R}_m - R_F}{\sigma_m} \right) \frac{\sigma_{im}}{\sigma_m} \tag{4}$$

where we use $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$.

CAPM: A More Rigorous Derivation

- Recall: an optimal condition for an efficient portfolio is

$$\bar{R}_k - R_F = \lambda \left(X_1 \sigma_{1k} + \ldots + X_N \sigma_{Nk} \right) \quad (5)$$

  where $\sigma_{ki} = Cov\left(R_k, R_i\right)$.

- Homogeneous expectation implies that the solution to this problem or the optimal portfolio here must be the market portfolio. As a result,

$$R_m = \sum_{i=1}^{N} R_i X_i \quad (6)$$

- Then, we can show that

$$X_1 \sigma_{1k} + \ldots + X_N \sigma_{Nk} = Cov\left(R_k, R_m\right). \quad (7)$$

CAPM: A More Rigorous Derivation

- So, we can write

$$\bar{R}_k - R_F = \lambda Cov\left(R_k, R_m\right) \tag{8}$$

- Since the market portfolio is one of an efficient portfolio, we can write

$$\bar{R}_m - R_F = \lambda Cov\left(R_m, R_m\right) = \lambda \sigma_m^2 \Longrightarrow \lambda = \frac{\bar{R}_m - R_F}{\sigma_m^2} \tag{9}$$

- Finally we can get the CAPM:

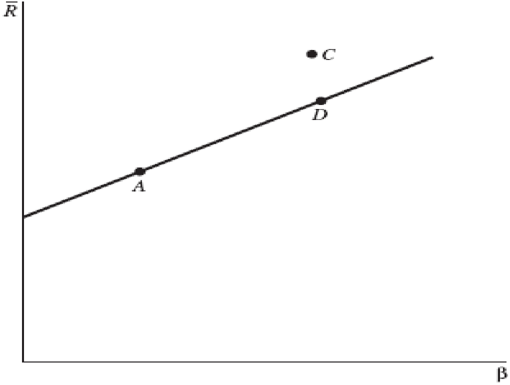
$$\begin{aligned} \bar{R}_k - R_F &= \frac{\bar{R}_m - R_F}{\sigma_m^2}\sigma_{km} = \left(\bar{R}_m - R_F\right)\frac{\sigma_{km}}{\sigma_m^2} \\ \bar{R}_i - R_F &= \left(\bar{R}_m - R_F\right)\beta_i \end{aligned} \tag{10}$$

Keys Assumptions for the Derivation of CAPM

1. No transaction costs, no personal income taxes.
2. Perfect competition.
3. Assets are infinitely divisible.
4. Investors have mean-variance preferences.
5. There exist a riskless asset.
6. Homogeneous expectation.

CAPM without Riskless: Simple Derivation

- Portfolios in expected return β space. This is the case without riskless asset.



CAPM without Riskless: Simple Derivation

- First, combinations of two risky portfolios lie on a straight line connecting them in expected return β space.
- Second, consider C and D. Using an arbitrage argument as before, we can show that all portfolios and securities must plot along the straight line, as in the figure.
- This line can be represented by

$$\bar{R}_i = a + b\beta_i \tag{11}$$

- Our job is to find what are a and b?

CAPM without Riskless: Simple Derivation

- Let $\bar{R}_Z$ be the expected return on a zero beta portfolio. This portfolio must also be on the straight line:

$$\bar{R}_Z = a + b \times 0 \Longrightarrow a = \bar{R}_Z \tag{12}$$

- Again, the market portfolio must be on the line as well:

$$\bar{R}_m = \bar{R}_Z + b \Longrightarrow b = \bar{R}_m - \bar{R}_Z \tag{13}$$

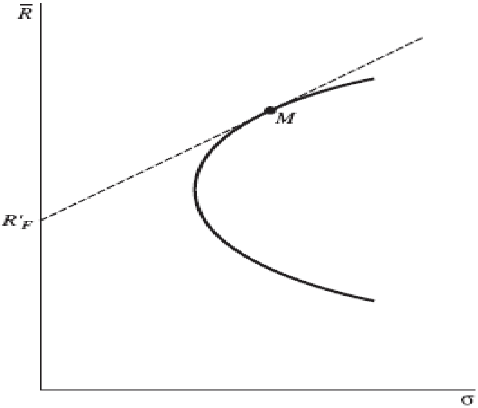
- Hence, we have an alternative CAPM (without riskless asset):

$$\bar{R}_i = \bar{R}_Z + \left(\bar{R}_m - \bar{R}_Z\right)\beta_i \tag{14}$$

CAPM without Riskless: A Rigorous Derivation

- Recall: an optimal condition for an efficient portfolio is

$$\bar{R}_k - R'_F = \lambda \left(X_1 \sigma_{1k} + \ldots + X_N \sigma_{Nk} \right) \quad (15)$$



CAPM without Riskless: A Rigorous Derivation

- Using the result we derived earlier,

$$X_1\sigma_{1k} + \ldots + X_N\sigma_{Nk} = Cov\left(R_k, R_m\right), \tag{16}$$

we can write

$$\bar{R}_k - R'_F = \lambda Cov\left(R_k, R_m\right) \tag{17}$$

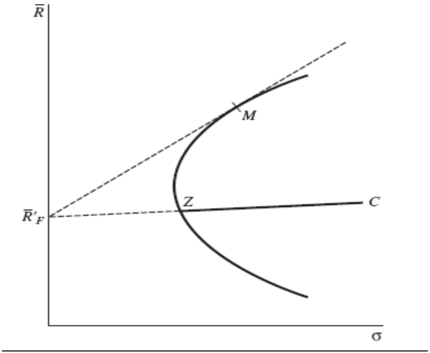
- We then can show that

$$\bar{R}_i = R'_F + \left(\bar{R}_m - R'_F\right)\beta_i \tag{18}$$

- Issue: This equation holds for any zero beta expected return R'_F. But we should use the least risky zero beta portfolio.

CAPM without Riskless: A Rigorous Derivation

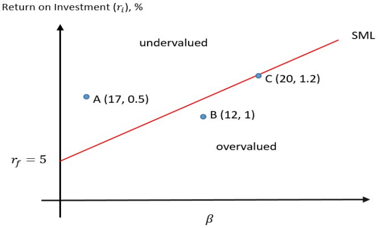
- The least risky zero beta portfolio is the zero beta portfolio that has the least total risk, the minimum variance zero beta portfolio Z.



- The CAPM now is

$$\bar{R}_i = \bar{R}_Z + \left(\bar{R}_m - \bar{R}_Z\right)\beta_i \tag{19}$$

Overvalued and undervalued



- Point A has the expected return which is above the SML (undervalued).
  - This asset has low level of risk (β) relative to amout of expected return.
  - This asset is a good choice for investment (buy).
- Point B has the expected return which is below the SML (overvalued).
  - Its expected return does not compensate for the level of risk (β).
  - This asset should not be considered for investment (sell).
- Point C has the expected return which is on the SML (properly valued).

Testing CAPM

Main Issues

- How can we test CAPM model?
- Are those tests reliable?

CAPM is in an Ex-ante Form

- The basic CAPM model can be written as

$$E\left(R_i\right) = R_F + \beta_i\left[E(R_m) - R_F\right] \tag{20}$$

- If lending and borrowing at the risk free rate is not possible or there is no risk free rate, then the CAPM becomes

$$E\left(R_i\right) = E(R_Z) + \beta_i\left[E(R_m) - E(R_Z)\right] \tag{21}$$

- Notice: these models are in an expectation form. This is suppose to be about future values.
- An expectation means that we are thinking about the situation before the uncertainty is realized. We call this ex-ante.

CAPM is tested using Ex-post Data

- On the other hand, we usually perform tests of CAPM models using realized (historical) data. These values are said to be ex-post values. How can we justify using ex-post data to test ex-ante model?
- Defense: using the law of large number, we should be able to estimate consistently (unbiased) the ex-ante expectation using sample mean of ex-post data.

Ex-post test of Ex-ante CAPM model

- That is, testing a CAPM model is a simultaneous test of all three following hypothesises:
  1. The market model holds
  2. The CAPM model holds
  3. β_i is stable over time
- NOTE: if there is no risk free rate, we will test the following model

$$\tilde{R}_{it} = \tilde{R}_{Zt} + \beta_i \left[\tilde{R}_{mt} - \tilde{R}_{Zt} \right] + \tilde{e}_{it} \tag{22}$$

Simple Test of CAPM

- Sharpe and Cooper (1972) divide stocks into ten portfolios using beta as a criterion. β at each point in time uses past 60 months of data. They also calculate average returns of each portfolio.
- They then estimate a linear equation

$$\bar{R}_i = 5.54 + 12.75\beta_i \tag{23}$$

with $R^2 > 0.95$.

- Key Points: the relationship between return and β is linear. Though the intercept (supposed to represent the risk free rate) is too high.

Simple Test of CAPM

- Lintner performed a similar but more systematic test of CAPM.
- First, run a time-series linear regression

$$R_{it} = \alpha_i + b_i R_{mt} + e_{it} \tag{24}$$

  using data of all stocks from 1954 to 1963.

- Then, run a cross-sectional regression

$$\bar{R}_i = a_1 + a_2 b_i + a_3 S_i + \epsilon_i \tag{25}$$

  where $S_i = Var(e_{it})$ is the residual variance, representing residual risk.

- The result shows that $a_3 = 0.237$ and statistically significant. This implies that CAPM is violated.

More Advanced Test of CAPM

- Miller and Scholes Test
  - If the CAPM is the right model, then β_i from the following equation should be consistent

$$\tilde{R}_{it} = (1 - \beta_i)R_{Ft} + \beta_i \tilde{R}_{mt} \tag{26}$$

  - If R_{Ft} is constant over time, there will no problem. But it is not the case in general. Hence, there should be an estimation biased.
  - Perhaps the relationship is nonlinear. But they found that it is not that important.
  - Heteroscedasticity. Again they found that it is not that important.
  - Measurement error in β_i. They found that this is significant.

More Advanced Test of CAPM

- Black, Jensen, and Scholes Test
  - They first run the following time-series regression

$$R_{it} - R_{Ft} = \alpha_i + \beta_i \left(R_{mt} - R_{Ft} \right) + e_{it} \qquad (27)$$

  - If CAPM generates the return data, then $\alpha = 0$.
  - They second run a cross-sectional regression of CAPM and found that

$$\bar{R}_i - R_F = 0.00359 + 0.01080\beta_i \qquad (28)$$

  - This result supports the CAPM.

Fama and MacBeth Test

- They use the same procedure as Black et al. to form 20 portfolios.
- The difference is in the cross-sectional regression:
  1. They run

$$\tilde{R}_{it} = \hat{\gamma}_{0t} + \hat{\gamma}_{1t}\beta_i - \hat{\gamma}_{2t}\beta_i^2 + \hat{\gamma}_{3t}S_{et} + \eta_{it} \tag{29}$$

  2. This regression is run each month, month by month. Therefore, they can study how the parameters change over time.

Fama and MacBeth Test

- This new form allow them to test the following hypotheses:
  1. $E(\hat{\gamma}_{3t}) = 0$: residual risk does not affect the return
  2. $E(\hat{\gamma}_{2t}) = 0$: test the linearity structure of CAPM
  3. $E(\hat{\gamma}_{1t}) = 0$: test the positivity of the price of risk
- If the first two hold, then we can conclude that a CAPM (either the standard or zero beta version) holds.
- Table 15.3 confirms that the first two hypothesises hold. In addition, they run a regression without those two terms and get better estimates.
- We then can conclude that residual risk has no effect. This is the opposite of Litner. the main reason is the measurement error. That is, using portfolios instead of securities reduce the error significantly (use the argument of Miller and Scholes).

Fama and MacBeth Test

- Given that a CAPM holds, then we can further distinguish between the standard and zero-beta CAPM using $E(\hat{\gamma}_{0t})$ and $E(\hat{\gamma}_{1t})$.
- If the zero beta model is the true model, the deviation of $\hat{\gamma}_{0t}$ from its mean $E(R_Z)$ and the deviation of $\hat{\gamma}_{1t}$ from its mean $E(R_m) - E(R_Z)$ must be random.
- Since we know that $E(R_Z) > R_F$, if $E(\hat{\gamma}_{0t} - R_F) > 0$ and $E(\hat{\gamma}_{1t} - E(R_m) + R_F) < 0$, we will then conclude that the zero beta model is the true model.
- They find that
  1. price of risk is positive,
  2. $\bar{\hat{\gamma}}_0$ is greater than R_F and $\bar{\hat{\gamma}}_1$ is less than $\bar{R}_m - R_F$
- These results support the zero beta CAPM.

Fama and MacBeth Test

- We can also test if the market operates as a fair game (efficient market). If CAPM is a true model, the expected value of $\hat{\gamma}_{2t}$ and $\hat{\gamma}_{3t}$ at time $t+1$ should be zero, regardless of past values.
- Fama and MacBeth test this implication by looking at the correlation of $\hat{\gamma}_{3t}$ with its lags values. They found that the correlation is not statistically different from zero. They also found a similar result for $\hat{\gamma}_{2t}$.
- They then conclude that the market operates as a fair game.

Fama and MacBeth Test of Thailand

Result for CAPM (Since 2001-2006)

2001-2006	$\bar{R}_{p\_347}$	$\bar{R}_{p\_20\_EqW}$	$\bar{R}_{p\_20\_MktW}$
$\hat{\gamma}_0$ (OLS) (MLE)	0.85899 (0.11051)*** (0.10986)***	1.33076 (0.37247)*** (0.33315)***	2.07639 (0.34058)*** (0.30462)***
$\hat{\gamma}_1$ (OLS) (MLE)	0.00011 (0.10764) (0.10701)	-0.28197 (0.43042) (0.38498)	-1.58736 (1.01708) **(0.90970)***
$\hat{\gamma}_2$ (OLS) (MLE)	0.00184 (0.00157) (0.00156)	-0.25425 (0.23090) (0.20652)	0.94942 (0.64644) (0.57820)
$\hat{\gamma}_3$ (OLS) (MLE)	0.11767 (0.00111)*** (0.00109)***	0.24791 (0.03363)*** (0.03008)***	0.01424 (0.04476) (0.04004)
$\bar{R}^2$	0.9921	0.9855	-0.0306

source: calculation

Note: * is significantly at 0.1 ** is significantly at 0.05 *** is significantly at 0.01

Fama and MacBeth Test of Thailand

Result for CAPM (Since 2009-2015)

2009-2015	$\bar{R}$p_449	$\bar{R}$p_20_EqW	$\bar{R}$p_20_MktW
$\hat{\gamma}_0$ (OLS) (MLE)	0.65727 (0.12657)*** (0.12601)***	0.58042 (0.26243)** (0.23472)**	0.43687 (0.47788) (0.42743)
$\hat{\gamma}_1$ (OLS) (MLE)	0.307467 (0.12552)** (0.12496)**	1.06995 (0.29325)*** (0.26229)***	1.42704 (0.63530)** (0.56823)**
$\hat{\gamma}_2$ (OLS) (MLE)	0.00933 (0.07807) (0.07772)	-0.41555 (0.19635)** (0.17562)**	-0.61852 (0.36940) **(0.33040)***
$\hat{\gamma}_3$ (OLS) (MLE)	0.11342 (0.00699)*** (0.00697)***	0.31225 (0.05360)*** (0.04794)***	0.20971 (0.13625) **(0.12187)***
$\bar{R}^2$	0.5642	0.7997	0.3680

source: calculation

Note: * is significantly at 0.1 ** is significantly at 0.05 *** is significantly at 0.01

CAPM is Not Testable?

- If any ex-post mean variance efficient portfolio p selected as the market portfolio, and β are computed using this portfolio as the market proxy, then

$$\bar{R}_i = \bar{R}_{ZP} + \beta_{ip} \left(\bar{R}_p - \bar{R}_{ZP} \right) \tag{30}$$

  must hold.
- That is, testing CAPM is not meaningful.

Roll's Proof

- Consider again the first order condition of an optimal portfolio problem:

$$\lambda \left(X_1 \sigma_{1k} + \ldots + X_N \sigma_{kN} \right) = \bar{R}_k - R_F \tag{31}$$

- Let p be the optimal portfolio, hence we can write

$$\lambda \sigma_{kp} = \bar{R}_k - R_F \tag{32}$$

  which must be true for any asset or portfolio.
- That is, it must be true for the optimal portfolio p as well:

$$\lambda \sigma_p^2 = \bar{R}_p - R_F \Rightarrow \lambda = \frac{\bar{R}_p - R_F}{\sigma_p^2} \tag{33}$$

- Hence, we have

$$\bar{R}_i = R_F + \frac{\sigma kp}{\sigma_p^2} \left(\bar{R}_p - R_F \right) = R_F + \beta_{kp} \left(\bar{R}_p - R_F \right) \tag{34}$$

Roll's Proof

- Using a similar argument as before, we can have a model without riskless:

$$\bar{R}_i = \bar{R}_{Zp} + \beta_{kp}\left(\bar{R}_p - \bar{R}_{Zp}\right) \quad (35)$$

  where $\bar{R}_{Zp}$ is the mean return of the minimum varaince zero beta portfolio.

- This proves that we can write a zero beta CAPM model with any efficient portfolio as a market proxy. But the true CAPM is the one with the true market portfolio.
- If we cannot observed the true market portfolio, then we cannot test the CAPM!