The Multi-Index Model

Tee Kilenthong
UTCC

Main Issues

- Can we predict covariance structure better by using more indices?
- How should we group assets, e.g. using industry or book to market value?

General Multi-Index Model

- The return of asset i is given by

$$R_i = a_i^* + b_{i1}^* I_1^* + \ldots + b_{iL}^* I_L^* + \epsilon_i^* \qquad (1)$$

  where I_l^* is the actual value of index l.
- In principle, I_l^* are correlated. We can get rid of the correlation using an orthogonalization technique, e.g., principal component.

$$R_i = a_i + b_{i1} I_1 + \ldots + b_{iL} I_L + \epsilon_i \qquad (2)$$

  where I_l is the uncorrelated or orthogonalized value of index l.
- Again this is time-series regression for asset by asset.

Orthogonalization

- Consider a two-index model

$$R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + c_i \quad (3)$$

where $Cov(I_1^*, I_2^*) \neq 0$.

- Set $I_1 = I_1^*$.
- Now run a (time-series) regression

$$I_2^* = \gamma_0 + \gamma_1 I_1 + d_t \quad (4)$$

Set

$$I_2 = d_t = I_2^* - \gamma_0 - \gamma_1 I_1 \Rightarrow I_2^* = I_2 + \gamma_0 + \gamma_1 I_1 \quad (5)$$

- Substituting I_1 and I_2 into the original equation

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + c_i \quad (6)$$

Regression Equation

- The equation for the return on a security can be written as

$$R_i = a_i + b_{i1} I_1 + \ldots + b_{iL} I_L + \epsilon_i \tag{7}$$

This is a linear regression equation.

- Key Assumptions:
  1. Error terms ϵ_i are unrelated/uncorrelated across assets:

$$E\left(\epsilon_i \epsilon_j\right) = 0, \text{ for all pairs } i, j, \tag{8}$$

  2. Covariance between an index I and ϵ_i is $E\left[\left(I_l - E\left(I_l\right)\right)\epsilon_i\right] = 0$.
  3. Why do not we need to assume this correlation assumption

$$E\left[\left(I_l - E\left(I_l\right)\right)\left(I_k - E\left(I_k\right)\right)\right] = 0 \tag{9}$$

Regression Equation

- By construction:
  1. Mean of ϵ_i is zero:
  $$E(\epsilon_i) = 0, \text{ for all stocks } i. \tag{10}$$
  2. Covariance between indices l and k is $E\left[\left(I_l - E\left(I_l\right)\right)\left(I_k - E\left(I_k\right)\right)\right] = 0.$
- By definition:
  1. Variance of ϵ_i is
  $$E(\epsilon_i)^2 = \sigma^2 = \sigma_{\epsilon_i}^2, \text{ for all stocks } i, \tag{11}$$
  2. Variance of I_l is
  $$E\left(I_l - E\left(I_l\right)\right)^2 = \sigma_l^2, \text{ for all stocks } i, \tag{12}$$

Covariance Structure

- The mean of a security i return is

$$E(R_i) = a_i + b_{i1}E(I_1) + \ldots + b_{iL}E(I_L) \quad (13)$$

- The Variance of a security i return is

$$\sigma_i^2 = b_{i1}^2\sigma_1^2 + \ldots + b_{iL}^2\sigma_L^2 + \sigma_{\epsilon_i}^2 \quad (14)$$

- The covariance of returns between securities i and j,

$$\sigma_{ij} = b_{i1}b_{j1}\sigma_1^2 + \ldots + b_{iL}b_{jL}\sigma_L^2 \quad (15)$$

Industry Index Models

- A general industry index model can be written as

$$R_i = a_i + b_{im} I_m + b_{i1} I_1 + \ldots + b_{iL} I_L + \epsilon_i \quad (16)$$

  where I_m is the uncorrelated or orthogonalized value of the market index, and I_j is an orthogonalized value of an industry index.

- A more constrained version

$$R_i = a_i + b_{im} I_m + b_{ij} I_j + \epsilon_i \quad (17)$$

  where asset i belongs to an industry j.

- This version assumes that an asset i in an industry j will not be affected by other industry factors.

Performances of Index Models

- How should we measure or compare performances of different models?
- Statistical test: we could judge a model using the differences between forecasted and actual correlation structures.
- More economic test: we could judge a model using the differences in return or profit that result from forecasts of each models.
- Elton and Gruber (1973) showed that adding more indexes reduces performances relative to the single index model.
- Cohen and Pogue (1967), using an industry index model, also showed that the single index model performs best.
- To summarize, there have been many studies and most of them showed that adding more indexes reduces performances relative to the single index model.

Chen-Roll-Ross Model

- Chen, Roll and Ross (1986) model is based on
  1. The value of share of stock depends on future cash flow. That is, factors affecting future cash flows and factors affecting discount rate shold determine price.
  2. Since current belief about those variables are incorporated in the prices only innovations or unexpected changes in those variables should determine return. That is, R^2 without the innovations should be significantly lower than otherwise.

Chen-Roll-Ross Model

- Example: Burmeister, et.al. (1988) use the following four factors determining prices:
  1. I_1: the return on long-term government bonds minus the return on long-term corporate bonds – default risk.
  2. I_2: the return on long-term government bonds minus the return on the one-month T-bill one month in the future – term structure.
  3. I_3: rate of inflation expected at the beginning of the month minus the actual rate of inflation realized at the end of the month – unexpected deflation.
  4. I_4: expected long-run growth rate in real final sales expected at the beginning of the month minus the expected long-run growth rate in real final sales expected at the end of the month– unexpected long-run profit.

Chen-Roll-Ross Model

- They then run

$$R_M - R_F = .0022 - 1.33I_1 + 0.56I_2 + 2.29I_3 - 0.93I_4$$

with $R^2 = 0.24$.

- The fifth index is the innovations

$$I_5 = (R_M - R_F) - (.0022 - 1.33I_1, +0.56I_2 + 2.29I_3 - 0.93I_4)$$

Chen-Roll-Ross Model

- Innovation Index is significant.

Table: Sector Sensitivities

	I_1	I_2	I_3	I_4	I_5	R^2
Sector Name						
Cyclical	-1.63	0.55	2.84	-1.04	1.14	0.77
Growth	-2.08	0.58	3.16	-0.92	1.28	0.84
Stable	-1.40	0.68	2.31	-0.22^a	0.74	0.73
Oil	-0.63^a	0.31	2.19^a	-0.83^a	1.14	0.50
Utility	-1.06	0.72	1.54	0.23^a	0.62	0.67
Transportation	-2.07	0.58	4.45	-1.13	1.37	0.66
Financial	-2.48	1.00	3.20	-0.56^a	0.99	0.72

a is not significant at 5%

Fama-French Model

- This type of models concerns with grouping of assets.
- Fama and French (1993) found that size and book-to-market (the ratio of book value to the market value) have a strong role in determining the cross section of average return on common stocks.
- Main Idea: size and book-to-market proxy common **risk** factors in returns.
- Small firms and low book-to-market are riskier than other firms (higher returns).
- Issue: book-to-market is available quarterly mostly but at least we will use monthly returns. How can we merge them?

Fama-French Model

- First, form 6 marketable portfolios using size (2 groups) and book-to-market (3 groups).
- Size:
  1. Big Firms: all stocks whose sizes are bigger than the median size.
  2. Smaller Firms: all stocks whose sizes are smaller than the median size.
- Book-to-Market value of equity: we will rank stocks
  1. Low BE/ME Firms: all stocks whose BE/ME are in the first 30 percentile.
  2. Medium BE/ME Firms: all stocks whose BE/ME are in between the 30 to 70 percentile.
  3. High BE/ME Firms: all stocks whose BE/ME are bigger than the 70 percentile.
- Note: both size and book-to-market are observed annually.

Fama-French Model

- Next, estimate/calculate **monthly** return of two portfolios out of those six portfolios in order to mimic risk factors.
  1. Small minus Big (SMB): the difference , each month, between the simple average of the returns on the three small-stock portfolio and the simple average of the returns on the three big-stock portfolio. This is to mimic the risk factor related to size.
  2. High minus Low (HML): the difference , each month, between the simple average of the returns on the two high-stock portfolio and the simple average of the returns on the two low-stock portfolio. This is to mimic the risk factor related to book-to-market equity value.
- Note: the returns of these two portfolios, or indexes, are monthly now.
- Fama and French (1993) showed that these two indexes help explain the variation in cross sectional monthly returns of stocks.

Fama-French Model

$$R_{it} - R_{Ft} = a_i + b_i(R_{Mt} - R_{Ft}) + s_i SMB_t + h_i HML_t + r_i RMW_t + c_i CMA_t + e_{it}$$

CAPM

Sharpe (1964), Lintner (1965) and Mossin (1966)

Three-Factor Model

Fama and French (1993)

Five-Factor Model

Fama and French (2014)

Fama-French Model

1. RMW: the difference, each month, between the simple average of the returns on the good (robust) profitability and the weak profitability. This is to mimic the risk factor related to profitability.

$$OP_{t-1} = \frac{(annual\ revenues - cost\ of\ goods\ sold - sell,\ general\ and\ administrative\ expense - interest\ expense)_{t-1}}{book\ equity_{t-1}} \tag{18}$$

2. CMA : the difference, each month, between the simple average of the returns on the low investment and the high investment. This is to mimic the risk factor related tohigh investment.

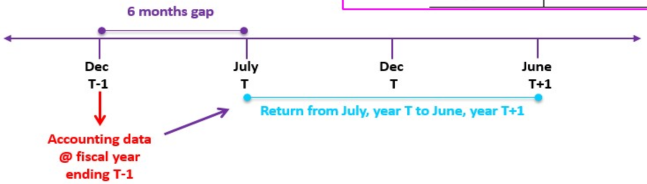
$$Inv_{t-1} = \frac{total\ asset_{t-1} - total\ asset_{t-2}}{total\ asset_{t-2}} \tag{19}$$

Fama-French Model

- 2x2 sort

	Breakpoint by Median	
	< Median	≥ Median
Size	S	B
B/M	L	H
OP	W	R
Inv	C	A

		Size	
		Small: S	**Big: B**
Size - B/M	Low	SL	BL
	High	SH	BH
Size - OP	Weak	SW	BW
	Robust	SR	BR
Size - Inv	Conservative	SC	BC
	Aggressive	SA	BA



Fama-French Model in Thailand

Table: Correlation metric

	Rm_Rf	SMB	HML	RMW	CMA
Rm_Rf	1				
SMB	-0.4955 (0)	1			
HML	0.29 (0.0024)	-0.0191 (0.845)	1		
RMW	0.256 (0.0078)	0.0776 (0.4272)	0.0839 (0.3903)	1	
CMA	-0.2627 (0.0063)	0.3579 (0.0002)	-0.0325 (0.7393)	-0.0511 (0.6013)	1

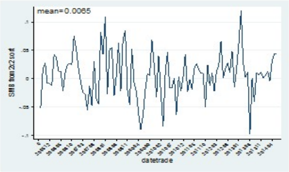
() is p-value.

Fama-French Model

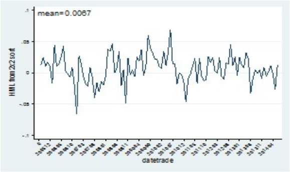
- 2x2 sort

Sort	Breakpoints	Factors and their Components
2 x 2 sorts on	Size: median	SMB = (SH + SL + SR + SW + SC + SA)/6 - (BH + BL + BR + BW + BC + BA)/6
Size and B/M, or	B/M: median	HML = (SH + BH)/2 - (SL+BL)/2
Size and OP, or	OP: median	RMW = (SR + BR)/2 - (SW+BW)/2
Size and Inv	Inv: median	CMA = (SC + BC)/2 - (SA+BA)/2

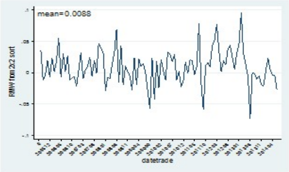
Fama-French Model in Thailand



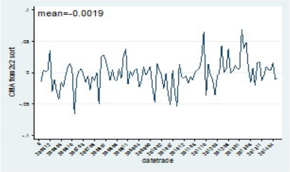
รูป 7 กราฟแสดงค่าตัวแปรปัจจัยขนาด (SMB) รายเดือนตั้งแต่สิงหาคม ค.ศ.2005 ถึงเดือนมิถุนายน ค.ศ.2014



รูป 8 กราฟแสดงค่าตัวแปรปัจจัยมูลค่า (HML) รายเดือนตั้งแต่สิงหาคม ค.ศ.2005 ถึงเดือนมิถุนายน ค.ศ.2014



รูป 9 กราฟแสดงค่าตัวแปรปัจจัยการทำกำไร (RMW) รายเดือนตั้งแต่สิงหาคม ค.ศ.2005 ถึงเดือนมิถุนายน ค.ศ.2014



รูป 10 กราฟแสดงค่าตัวแปรปัจจัยการลงทุน (CMA) รายเดือนตั้งแต่สิงหาคม ค.ศ.2005 ถึงเดือนมิถุนายน ค.ศ.2014

Fama-French Model in Thailand

5 - Factor							
กลุ่มหลักทรัพย์	a	b	s	h	r	c	Adj.R^2
BA	0.0001	0.9652	-0.0818	-0.1043	-0.0296	-0.2767	97.4%
	(0.045)	(46.819)*	(-2.562)**	(-2.172)**	(-0.745)	(-5.692)*	
BC	0.0000	0.8665	-0.0680	0.1137	-0.1628	0.6522	91.2%
	(-0.008)	(26.172)*	(-1.326)	(1.474)	(-2.556)**	(8.355)*	
BH	-0.0024	0.9933	0.1246	0.5698	-0.1499	-0.0143	94.1%
	(-1.400)	(30.904)*	(2.503)**	(7.612)*	(-2.424)**	(-0.189)	
BL	-0.0005	0.9082	-0.1159	-0.1112	-0.0553	0.0690	97.7%
	(-0.487)	(51.424)*	(-4.237)*	(-2.704)*	(-1.627)	(1.658)	
BR	0.0004	0.9421	-0.1526	-0.0945	0.1120	0.0318	97.4%
	(0.357)	(46.419)*	(-4.855)*	(2.000)**	(2.868)*	(0.666)	
BW	-0.0017	0.9683	0.1614	0.2917	-0.4975	0.0004	83.3%
	(-0.605)	(18.719)*	(2.014)**	(2.421)**	(-4.999)*	(0.004)	
SA	-0.0012	0.9209	1.0369	0.3534	-0.1002	-0.4381	93.6%
	(-0.791)	(32.290)*	(23.472)*	(5.320)*	(-1.826)***	(-6.515)*	
SC	-0.0012	1.0196	1.0231	0.1355	0.0330	0.6329	94.4%
	(-0.746)	(35.605)*	(23.066)*	(2.031)**	(0.600)	(9.374)*	

Fama-French Model in Thailand

ตารางที่ 3 แสดงตารางสรุปทางสถิติของค่าสัมประสิทธิ์การตัดสินใจที่ปรับแล้ว (Adjusted R^2) ของแบบจำลองราคาห้าปัจจัย แบบจำลองราคาสามปัจจัย และแบบจำลอง CAPM ในการอธิบายอัตราผลตอบแทนเฉลี่ยส่วนเกินของ 12 กลุ่มหลักทรัพย์ที่จัดแบบ 2x2 ตามวิธีการของ Fama-French ในช่วงเดือนกรกฎาคม ค.ศ.2003 ถึงเดือนมิถุนายน ค.ศ. 2014

	Adj.R^2 of 5-Factor	Adj.R^2 of 3-Factor	Adj.R^2 of CAPM
Mean	0.9277	0.8853	0.6940
S.D.	0.0460	0.1124	0.2295
Min	0.8326	0.5753	0.4182
Max	0.9773	0.9765	0.9702
Note	7 จาก 12 กลุ่มหลักทรัพย์มีค่า Adj.R^2 เท่ากับค่าเฉลี่ย Adj.R^2 ขึ้นไป	8 จาก 12 กลุ่มหลักทรัพย์มีค่า Adj.R^2 เท่ากับค่าเฉลี่ย Adj.R^2 ขึ้นไป	6 จาก 12 กลุ่มหลักทรัพย์มีค่า Adj.R^2 เท่ากับค่าเฉลี่ย Adj.R^2 ขึ้นไป

Arbitrage Pricing Theory (APT)

- What is APT?
- Can APT perform better than CAPM?
- Can we test APT against CAMP?

Arbitrage Pricing Theory (APT): Ross (1976)

- APT is based on a multi-index model, which we discussed before:

$$R_i = a_i + b_{i1} I_1 + \ldots + b_{iJ} I_J + \varepsilon_i \quad (20)$$

  where I_j is the value of the j^{th} index, and b_{ij} is the sensitivity of stock i's return to the j^{th} index.
- Assumptions: $E[\varepsilon_i \varepsilon_\ell] = 0$ for all i, ℓ where $i \neq \ell$, and $E[\varepsilon_i (I_j - E[I_j])]$.
- Key Idea: APT is based on the law of One Price: two items that are "the same" cannot be sold at different prices.

Key Assumptions of APT

- Investors' expectations are homogeneous.
- Residual risk tends to go to zero; that is, investors hold a well diversified portfolio.
- Accordingly, only systematic risks through indexes matter for return. In addition, b_{ij} measures the impact of each factor j on the return of each asset i.
- These are not much different from CAPM, however.
- The main different is that APT allows for more factors that can affect asset return while the market is the only factor that matters for CAPM.

A Simple Derivation of APT

- Consider the following two-index model:

$$R_i = a_i + b_{i1} I_1 + b_{2i} I_2 + \varepsilon_i \tag{21}$$

where $E\left(\varepsilon_i \varepsilon_{ell}\right) \approx 0$.

- There are following three portfolios:

Portfolio	Expected Return	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- These three portfolios form a plane describing by the following equation

$$\bar{R} = 7.75 + 5 b_{i1} + 3.75 b_{i2} \tag{22}$$

A Simple Derivation of APT

- Suppose that there is also a protfolio E with $E(R_E) = 15\%$, and $b_{E1} = 0.6, b_{E2} = 0.6$.
- On the other hand, if we substitute $b_{E1} = 0.6, b_{E2} = 0.6$ into the return equation, we will get $\bar{R} = 13\%$. What does it mean?
- This implies an arbitrage opportunity. How?
- Hint: this is similar to the argument we used to prove CAPM before.

A Simple Derivation of APT

- To see the arbitrage in detailed, let form a portfolio (called portfolio D) from the original three protfolios as follows. Let $X_i = \frac{1}{3}$ for $i = A, B, C$ (equally weighted portfolio).
- As a result,

$$
\begin{aligned}
b_{D1} &= \sum_{i=A,B,C} X_i b_{i1} = \frac{1}{3} \times 1.0 + \frac{1}{3} \times 0.5 + \frac{1}{3} \times 0.3 = 0.6, \\
b_{D2} &= \sum_{i=A,B,C} X_i b_{i2} = \frac{1}{3} \times 0.6 + \frac{1}{3} \times 1.0 + \frac{1}{3} \times 0.2 = 0.6, \\
\bar{R}_D &= \sum_{i=A,B,C} X_i \bar{R}_i = \frac{1}{3} \times 15 + \frac{1}{3} \times 14 + \frac{1}{3} \times 10 = 13.
\end{aligned}
$$

A Simple Derivation of APT

- What we have now are two portfolios with the same factors, i.e., $b_{D1} = b_{E1}$ and $b_{D2} = b_{E2}$ but different expected returns, $\bar{R}_D < \bar{R}_E$.
- An investor then can arbitrage by long D and short E:

	Investment	Expected Return	b_{i1}	b_{i2}
D	+100	-113	-0.6	-0.6
E	-100	+115	0.6	0.6
Arbitrage Porfolio	0	2	0	0

- Earn positive expected return without any “systematic risk” and no investment required.
- Using a similar argument as in CAPM (no arbitrage opportunity in equilibrium), we can show that, in equilibrium

$$\bar{R} = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{2i}. \quad (23)$$

What does it mean geographically?

A More Rigorous Derivation of APT

- This way we can see assumptions/conditions of APT clearer.
- A sufficient condition for APT to hold is that there are enough securities in the market that it is possible to form a protfolio with the following property:

$$\sum_{i=1}^{N} X_i b_{ij} = 0, \text{ for all } j = 1, \ldots, J, \tag{24}$$

$$\sum_{i=1}^{N} X_i = 1, \tag{25}$$

$$\sum_{i=1}^{N} X_i \varepsilon_i = 0, \tag{26}$$

where the first one implies that the portfolio has no systematic risk.

A More Rigorous Derivation of APT

- Then, *No Arbitrage Condition* implies that

$$\sum_{i=1}^{N} X_i \bar{R}_i = 0. \tag{27}$$

- To sum up, (24) and (25) mathematically imply that $\mathbf{X}$ is orthogonal to vectors of b_{ij} for each j and orthogonal to a vector of ones, respectively.
- Then, condition (27) requires that such $\mathbf{X}$ must be orthogonal to the vector of expected returns $\bar{\mathbf{R}}$.

A More Rigorous Derivation of APT

- We then can prove mathematically that we can write the last vector (the vector of expected returns) as a *linear* combination of vectors of b_{ij} and a vector of ones, i.e.,

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \ldots + \lambda_J b_{iJ} \tag{28}$$

Methodology for Testing APT

- There are 4 approaches have been used to test APT.
  1. Factor Analysis.
  2. Asset Characteristics as Factors.
  3. Macroeconomic and Market Variables as Factors.
  4. Portfolio Characteristics as Factors.

Factor Analysis Approach for Testing APT

- Recall: there are two main equations:

$$R_i = a_i + \sum_{j=1}^{J} b_{ij} I_j + \varepsilon_i, \quad (29)$$

$$\bar{R}_i = R_F + \sum_{j=1}^{J} \lambda_j b_{ij}. \quad (30)$$

where I_j is the j^{th} *factor*, b_{ij} is the *factor loading* for asset i with respect to the j^{th} factor, and λ_j is the *market price of risk* associated with the j^{th} factor.

Factor Analysis Approach for Testing APT

- Factor Analysis Approach assumes that both I_j and b_{ij} are unknown and will be estimated simultaneously.
- Key Idea: the factor I_j is common across assets while the factor loading b_{ij} is asset specific. Factor Analysis is to minimize the covariance of the residual returns.
- Advantage: the data will tell us how many factors (we have to subjectively decide which one is the last factor though). No need to assume a priori.
- Problems: there are a few problems associated with Factor Analysis.
  1. No meanings to the signs of the factors.
  2. The scales of I_j and b_{ij} are arbitrary.
  3. The order of the factors are potentially different across different samples.

Factor Analysis Approach for Testing APT

- Roll and Ross (1980) applies factor analysis to estimate APT.
  1. They applied factor analysis with 42 groups of 30 stocks.
  2. There seem to be 5 factors that matter for the return-generating equation (29).
  3. There are at least 3 factors that are significant in explaining equilibrium prices, see (30).
  4. However, this can not tell if (zero-beta) CAPM is rejected?

Factor Analysis Approach for Testing APT

- Cho, Elton,and Gruber (1984).
  1. They applied the same methodology as in Roll and Ross (1980) to a later period.
  2. They then simulated a set of data using the zero-beta CAPM while enforcing the same means and variances on the returns for each stock.
  3. They then applied the same methodology to the simulated data to check whether the methodology can point to the zero-beta CAPM when it is the true model.
  4. The answer is YES. Therefore, they can conclude that there are additional factors beyond those in the zero-beta CAPM determine equilibrium prices.
  5. This seems to suggest that APT fits the data better than CAPM.

Factor Analysis Approach for Testing APT

- However, those are not really a test of APT yet.
- In fact, it is much more difficult to test APT relative to CAPM. This is the cost of being more flexible.
- Nonetheless, APT implies that if the model is correct and the factor analysis correctly identify factors, then the market price of risk λ_0 and λ_j should be the same for each groups.
  1. Brown and Weinstein (1983) tested these implications. Their results are at best inconclusive (a little bit against APT).
  2. Dhrymes, Friend, and Gultekin (1984) also found an ambiguous result.
- Overall: testing results based on factor analysis are inconclusive. We can perhaps only say that it seems to suggest that more than two factors are priced.

Using Asset Characteristics as Factors for Testing APT

- This is very much similar to Fama and MacBeth (1973) in that it adds additional variables rather than just beta.
- Sharpe (1982) tried this by adding size, Bond Alpha, etc.
- He found that size and other variables seem to be significant.

Using Macroeconomic and Market Variables as Factors for Testing APT

- Chen, Roll, and Ross (1986) assumed that Inflation, The term Structure of Interest Rate (the difference between the returns of bonds with long and short maturity), Risk Premia (difference between the returns of different classes of assets (AAA versus BAA), and Industrial Production are Factors.
- They then tested if
  1. these factors are correlated with the factors from factor analysis. Answer: YES,
  2. whether these help explain equilibrium prices.
- They found that
  1. the market priced all these factors if we exclude beta,
  2. but market prices of risk of those factors are not significant once they included beta.
  3. This does not support APT much. But the problem perhaps is that the factors they chosen are not the right ones.

Using Macroeconomic and Market Variables as Factors for Testing APT

- Burmeister and McElroy (1988) picked four observable factors including,
    1. default risk (difference between government and corporate bonds),
    2. time premium (difference between long term and short term government bonds),
    3. deflation (expected inflation - the real one),
    4. change in expected sales.

Using Macroeconomic and Market Variables as Factors for Testing APT

- Burmeister and McElroy (1988) also define the fifth (unobserved) factor based on the first four factors.
  1. This factor is derived from a well-diversified portfolio called m and the first four factors:

$$R_{mt} = \lambda_m + R_{Ft} + \sum_{j=1}^{4} b_{mj} F_{jt} + \varepsilon_{mt} \tag{31}$$

They assumed that the fifth factor is the error term, i..e, $F_{kt} = \varepsilon_{mt}$. Therefore, it can be estimated as follows:

$$\hat{F}_{kt} = (R_{mt} - R_{Ft}) - \left[\lambda_m + \sum_{j=1}^{4} b_{mj} F_{jt} \right]. \tag{32}$$

The pricing equation is now

$$R_{it} = R_{Ft} + \sum_{j=1}^{4} b_{ij} F_{jt} + b_{ik} \hat{F}_{kt} + \varepsilon_{mt} \tag{33}$$

Using Macroeconomic and Market Variables as Factors for Testing APT

- Burmeister and McElroy (1988) found that all factors are priced in equilibrium; that is, λ_j are significantly different from zero for all $j = 1, 2, 3, 4, 5$.
- Burmeister and McElroy (1988) conclude that CAPM can be rejected a t 1% significance level.
- On the other hand, they cannot reject APT.

Using Portfolio Characteristics as Factors for Testing APT

- This is what Fama and French (1993) is about.
- They formed 6 marketable portfolios using size (2 groups) and book-to-market (3 groups).
- Size:
  1. Big Firms: all stocks whose sizes are bigger than the median size.
  2. Smaller Firms: all stocks whose sizes are smaller than the median size.
- Book-to-Market value of equity: we will rank stocks
  1. Low BE/ME Firms: all stocks whose BE/ME are in the first 30 percentile.
  2. Medium BE/ME Firms: all stocks whose BE/ME are in between the 30 to 70 percentile.
  3. High BE/ME Firms: all stocks whose BE/ME are bigger than the 70 percentile.

Using Portfolio Characteristics as Factors for Testing APT

- They tested whether the intercepts of the time series of excess returns equals to zero as APT would suggest?
- They found that the intercepts are indeed zero.
- They concluded that five factors do a good job explaining common variations in bond and stock returns, and the cross-section of average returns.

Can we really test APT against Zero-beta CAPM?

- The answer is NOT QUITE.
- Consider the following return-generating function

$$R_i = a_i + \beta_i R_m + \varepsilon_i \tag{34}$$

- If a riskless exists, then we have

$$R_i = R_F + \beta_i \left(\bar{R}_m - R_F \right) \tag{35}$$

- Now suppose the return generating function as a multi-index model is

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + \varepsilon_i \tag{36}$$

- The APT model is

$$\bar{R}_i = R_F + b_{i1} \lambda_1 + b_{i2} \lambda_2 \tag{37}$$

Can we really test APT against Zero-beta CAPM?

- Recall that we can interpret λ_j as excess return of a portfolio i with $b_{ij} = 1$ and zero for other asset.
- Suppose again that the true model is the CAPM. This must be true for any asset and any portfolio. Therefore, we can run it on the market return and get its $\beta_{\lambda j}$ as follows:

$$\lambda_1 = \beta_{\lambda 1}\left(\bar{R}_m - R_F\right), \quad (38)$$

$$\lambda_2 = \beta_{\lambda 2}\left(\bar{R}_m - R_F\right) \quad (39)$$

- Substituting these conditions into APt equation gives

$$\bar{R}_i = R_F + \left(b_{i1}\beta_{\lambda 1} + b_{i2}\beta_{\lambda 2}\right)\left(\bar{R}_m - R_F\right) = R_F + \beta_i\left(\bar{R}_m - R_F\right)$$

- APT solution with multiple factors is fully consistent with CAPM.
- Finding more factors explains covariance between security returns does not imply that CAPM fails.