Weiner Processes and Itô's Lemma

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Main Issues

- Brownian Motion
- Itô's Lemma: Stochastic version of Taylor approximation.

Markov Property

- A Markov process is a stochastic process where only the current values of variables are relevant for predicting the future.
- For example, let $\mathbf{x}_t$ be a random vector, which are realized in period t. That is, we can observe their values at time t.
- It is Markov process if

$$E\left[\mathbf{x_{t+1}} | \mathbf{x_t}, \mathbf{x_{t-1}}, \mathbf{x_{t-2}}, ...\right] = E\left[\mathbf{x_{t+1}} | \mathbf{x_t}\right] \tag{1}$$

- Main advantage is that we have to keep track only one lag.

Weiner Processes

- A discrete-time random walk, x_t, is

$$x_t = x_{t-1} + \epsilon_t \tag{2}$$

where ϵ_t is i.i.d. discrete random variable. For example, a random walk (without a drift) has $Prob(\epsilon_t = 1) = Prob(\epsilon_t = -1) = \frac{1}{2}$. This means that the mean of x_t is zero.

Weiner Processes

- A Weiner process, $z(t)$, is a stochastic process (a continuous time version of random walk) with
  1. it is a Morkov process,
  2. it has independent increments,
  3. changes over any finite interval of time are normally distributed.
- More formally, condition (3) implies that

$$\Delta z = \epsilon_t \sqrt{\Delta t}, \tag{3}$$

  where $\epsilon_t \sim N(0,1)$.
- Condition (2) implies that

$$E\left[\epsilon_t \epsilon_s\right] = 0, fort \neq s \Rightarrow \epsilon_t = \epsilon \tag{4}$$

Weiner Processes

- More properties:

$$E\left[z(T) - z(0)\right] = E\left[\sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}\right] = 0 \tag{5}$$

where $N = \frac{T}{\Delta t}$.

$$\begin{aligned} Var\left[z(T) - z(0)\right] &= Var\left[\sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}\right] \\ &= \sum_{i=1}^{N} \Delta t Var\left(\epsilon_i\right) = N\Delta t = T \end{aligned} \tag{6}$$

Weiner Processes with Drift: Differential Form

- A Weiner process with drift, $x(t)$, can be written in differential form as

$$dx = \alpha dt + \sigma dz, \tag{7}$$

where z is the standard Weiner process, α is the drift parameter, and σ is the variance parameter.

- Properties:

$$E\left[dx\right] = E\left[\alpha dt + \sigma dz\right] = \alpha dt \Rightarrow E\left[x(t)\right] = \alpha t, \tag{8}$$

Weiner Processes with Drift: Differential Form

- More Properties:

$$
\begin{aligned}
Var\left[dx\right] &= E\left[(dx)^2\right] - E\left[dx\right]^2 \\
&= E\left[\alpha^2(dt)^2 + 2\alpha\sigma dtdz + \sigma^2(dz)^2\right] - \left[\alpha dt\right]^2 \\
&= \alpha^2(dt)^2 + \sigma^2 E\left[(dz)^2\right] - \alpha^2(dt)^2 \\
&= \sigma^2 dt \Rightarrow Var\left[x(t)\right] = \sigma^2 t
\end{aligned} \tag{9}
$$

Ex:Binomial model

Example: We look at a binomial model.

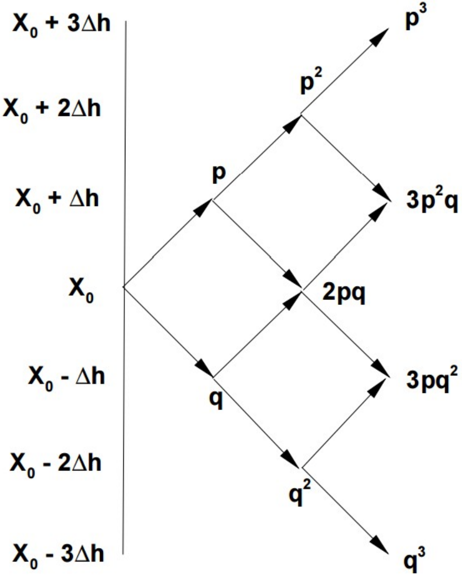
$$\begin{array}{cccl} \text{period-0} & & \text{period-1} & \\ x_0 & \rightarrow & X_1 = x_0 + \Delta h & \text{with probability } p \\ x_0 & \rightarrow & X_1 = x_0 - \Delta h & \text{with probability } q \end{array} \tag{10}$$

(11)

where

- $p + q = 1$
- X can move only up or down Δh.
- The probability of an up or down is independent of what happened in the past.

Ex:Binomial model



Ex:Binomial model

Let

$$\begin{aligned} \Delta X &= X_1 - X_0 = \Delta h \quad \text{with probability } p \\ \Delta X &= X_1 - X_0 = -\Delta h \quad \text{with probability } q \end{aligned} \tag{12}$$

Then

$$E(\Delta X) = (p-q)\Delta h \tag{13}$$

$$E((\Delta X)^2) = p(\Delta h)^2 + q(-\Delta h)^2 = (\Delta h)^2 \tag{14}$$

and,

$$Var(\Delta X) = E((\Delta X)^2) - E(\Delta X)^2 \tag{15}$$

$$= (\Delta h)^2 - (p-q)^2(\Delta h)^2 = 4pq(\Delta h)^2 \tag{16}$$

Ex:Binomial model

We consider the distribution of X after " n moves" ,so that $t = n\Delta t$. The probability of j up moves ($P(n,j)$) is

$$P(n,j) = \frac{n!}{j!(n-j)!} p^j q^{n-j} \tag{17}$$

if X_n is the value of X after n steps on the lattice, then

$$E(X_n - X_0) = nE(\Delta X) \tag{18}$$

$$Var(X_n - X_0) = nVar(\Delta X) \tag{19}$$

from (13) , (16) and $n = \frac{t}{\Delta t}$

$$E(X_n - X_0) = n(p-q)\Delta h = \frac{t}{\Delta t}(p-q)\Delta h \tag{20}$$

$$Var(X_n - X_0) = nVar(\Delta X) = \frac{t}{\Delta t} 4pq(\Delta h)^2 \tag{21}$$

Ex:Binomial model

- We have to choose $\Delta h = \sigma\sqrt{\Delta t}$
- If you don't choose $\Delta h = constant * \sqrt{\Delta t}$ so, $Var(X_n - X_0) \to 0$ or ∞ where $\Delta t \to 0$

We get

$$E(X_n - X_0) = \frac{t}{\Delta t}(p-q)\sigma\sqrt{\Delta t} = \frac{(p-q)\sigma t}{\sqrt{\Delta t}} \quad (22)$$

$$Var(X_n - X_0) = \frac{t}{\Delta t}4pq(\Delta h)^2 = 4pqt\sigma^2 \quad (23)$$

Ex:Binomial model

- For $E(X_n - X_0)$ to be independent of Δt as $\Delta t \to 0$ then $p - q = constant * \sqrt{\Delta t}$
- We choose $p - q = \frac{\alpha}{\sigma}\sqrt{\Delta t}$ so

$$p = \frac{1}{2}(1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}) \quad (24)$$

$$q = \frac{1}{2}(1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}) \quad (25)$$

so, we get

$$E(X_n - X_0) = \alpha t \quad (26)$$

$$Var(X_n - X_0) = \sigma^2 t(1 - \frac{\alpha^2}{\sigma^2}\Delta t) = \sigma^2 t \ ; \ \Delta t \to 0 \quad (27)$$

Ex:Binomial model

Let $X(t_n) - X(t_0) = X_n - X_0$ is very small, so that $X_n - X_0 \simeq dX$ and $t_n - t \simeq dt$ and

$$E(dX) = \alpha dt \tag{28}$$

$$Var(dX) = \sigma^2 dt \tag{29}$$

- We agree with (8), (9) . Hence, in the limit $\Delta t \to 0$
- We can write the solution to the stochastic differential equation(SDE)

Ex:Binomial model

$$dX = \alpha dt + \sigma dZ \tag{30}$$

where

$$dZ = \phi\sqrt{dt}, \tag{31}$$

where ϕ is a random variable. We assume $\alpha = 0, \sigma = 1$ then

$$dX = dZ \simeq Z(t_i) - Z(t_{i-1}) = Z_i - Z_{i-1} = X_i - X_{i-1} \tag{32}$$

and

$$E(Z_n - Z_0) = 0 \tag{33}$$

$$Var(Z_n - Z_0) = dt \tag{34}$$

If n is large $(\Delta t \to 0)$, recall that the binomial distribution (17) tends to a normal distribution

$$\int_0^t dZ = (Z_n - Z_0) \sim N(0, t) \tag{35}$$

Ex:Binomial model

By construction (see figure describing the binomial model above)

$$\begin{aligned} \Delta Z &= Z_i - Z_{i-1} = \sqrt{\Delta t} \qquad \text{with probability } p \\ \Delta Z &= Z_i - Z_{i-1} = -\sqrt{\Delta t} \qquad \text{with probability } q \end{aligned} \tag{36}$$

then $(Z_i - Z_{i-1})^2 = \Delta t$ with certainty, so we can write

$$(Z_i - Z_{i-1})^2 = (dZ)^2 = \Delta t \tag{37}$$

Importantly, we now have that square of a Weiner process is not even random, i.e. it is a constant Δt.

A Generalized Weiner Process: Ito's Process

- An Ito's process is

$$dx = \alpha(x,t)\,dt + \beta(x,t)\,dz \tag{38}$$

- Geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz \Rightarrow \frac{dS}{S} = \mu dt + \sigma dz \tag{39}$$

This is a popular stochastic process representing a stock price.

- Its discrete-time counter part is given by S such that

$$\frac{\Delta S}{S} \sim N\left(\mu \Delta t, \sigma^2 \Delta t\right) \tag{40}$$

Geometric Brownian motion: Example

- Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum.
- That means

$$\mu = 0.15, \sigma = 0.30 \tag{41}$$

- The stock process (as a Geometric Brownian motion) is

$$\frac{\Delta S}{S} = 0.15dt + 0.30dz \tag{42}$$

- Its discrete-time approximation process for a time interval of a week is

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t} \Rightarrow \frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.3 \times 0.13867\epsilon$$

Correlated Processes

- Consider two processes

$$\begin{aligned} dx_1 &= \alpha_1 dt + \sigma_1 dz_1, \\ dx_2 &= \alpha_2 dt + \sigma_2 dz_2 \end{aligned}$$

  where $E(dz_1 dz_2) = \rho dt$.

- We can approximate these processes by

$$\begin{aligned} dx_1 &= \alpha_1 dt + \sigma_1 u\sqrt{\Delta t}, \\ dx_2 &= \alpha_2 dt + \sigma_2(\rho u + \sqrt{1-\rho^2}v)\sqrt{\Delta t} \end{aligned}$$

  where u, v are standard normal distributed random variables.

Ito's Lemma

- Ito's Lemma helps approximate a stochastic process that is a function of underlying stochastic processes.
- Consider a process

$$dx = \alpha(x,t)\,dt + \beta(x,t)\,dz \tag{43}$$

- Ito Lemma is based on standard Taylor approximation with the only exception that we need to keep track up to the second order. This is because the Weiner process is proportional to $\sqrt{dt}$, and we have to keep dt. That is, we can cancel out only the dt terms with the order strictly higher than 1.

Ito's Lemma

- Consider a function $G(x,t)$. Question: what is the process of this function given that we know the process of x.
- Total derivative of G gives

$$
\begin{aligned}
dG &= \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(dx)^2 \\
&+ \frac{\partial^2 G}{\partial x \partial t}dxdt + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(dt)^2 \\
&= \left[\frac{\partial G}{\partial x}\alpha + \frac{\partial G}{\partial t}\right]dt + \frac{\partial G}{\partial x}\sigma dz + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(dx)^2
\end{aligned}
$$

Ito's Lemma

- Consider

$$
\begin{aligned}
(dx)^2 &= \alpha^2(dt)^2 + 2\alpha\sigma\epsilon dt\sqrt{dt} + \sigma^2\epsilon^2 dt \\
&\approx \sigma^2\epsilon^2 dt
\end{aligned}
$$

- Thus the total derivative is now

$$
dG = \left[\frac{\partial G}{\partial x}\alpha + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2\epsilon^2\right] dt + \frac{\partial G}{\partial x}\sigma dz
$$

- We usually take expectation, we can therefore write this equation as

$$
dG = \left[\frac{\partial G}{\partial x}\alpha + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2\right] dt + \frac{\partial G}{\partial x}\sigma dz \qquad (44)
$$

where we use $E[\epsilon^2] = 1$.

Ito's Lemma: Geometric Brownian motion case

- We now consider a geometric Brownian motion process

$$dS = \mu S dt + \sigma S dz \tag{45}$$

- This case we have

$$dG = \left[\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right] dt + \frac{\partial G}{\partial S} \sigma S dz \tag{46}$$

Ito's Lemma: Forward Contracts

- The relationship between forward price F and stock price S is

$$F = Se^{r(T-t)} \tag{47}$$

- Using Ito's Lemma, we can get the process of the forward price as

$$dF = (\mu - r)Fdt + \sigma Fdz \tag{48}$$

Note that it is a geometric Brownian motion.

Ito's Lemma: log-normality

- Suppose that S follow the geometric Brownian motion as specified earlier. In fact, it means that S is log-normally distributed.
- Using Ito's Lemma, we can get the process of the normal random variable, $G = \ln S$:

$$dG = (\mu - \frac{\sigma^2}{2})dt + \sigma dz \tag{49}$$

- That is, if G is normal with mean $\mu - \frac{\sigma^2}{2}$ and variance σ^2, S will have mean μ and variance σ^2.
- If x is normal with mean μ and variance σ^2, its log-normal counterpart has mean $\mu + \frac{\sigma^2}{2}$ and variance σ^2

Ito's Lemma: with correlated processes

- Consider two processes

$$\begin{aligned} dx_1 &= \alpha_1 dt + \sigma_1 dz_1, \\ dx_2 &= \alpha_2 dt + \sigma_2 dz_2 \end{aligned}$$

  where $E(dz_1 dz_2) = \rho dt$.
- Let $G(x_1, x_2)$ be a function of x_1 and x_2.
- Ask students to show that

$$\begin{aligned} dG &= \left[\frac{\partial G}{\partial x_1}\alpha_1 + \frac{\partial G}{\partial x_2}\alpha_2 + \frac{\partial G}{\partial t}\right] dt \\ &+ \left[\frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma_1^2 + \frac{\partial^2 G}{\partial x_1 \partial x_2}\sigma_1\sigma_2\rho + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma_2^2\right] dt \\ &+ \frac{\partial G}{\partial x_1}\sigma_1 dz_1 + \frac{\partial G}{\partial x_2}\sigma_2 dz_2 \end{aligned}$$