Black-Scholes-Merton Model

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Main Issues

- Black-Scholes-Merton PDE.
- Black-Scholes-Merton Formula for a European option.

Lognormal Perperty of Stock Prices

- Consider the stock price process

$$dS = \mu S dt + \sigma S dz \Rightarrow \frac{dS}{S} = \mu dt + \sigma dz \tag{1}$$

- This implies that

$$\frac{\Delta S}{S} \sim N\left(\mu \Delta t, \sigma^2 \Delta t\right) \tag{2}$$

- Using Ito's lemma, we can show that (by integration)

$$\ln S_t - \ln S_0 \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right) \tag{3}$$

and

$$\ln S_t \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right) \tag{4}$$

Lognormal Perperty of Stock Prices: Example

- Consider a stock with an initial price \$40, an expected return (rate) of 16% per annum, and a volatility of 20 % per annum.
- Using the equation before, the probability of stock price in 6 months ($T = 0.5$) from today is

$$\begin{aligned} \ln S_T &\sim N\left(\ln 40 + \left(0.16 - \frac{0.20^2}{2}\right) \times 0.5, 0.20^2 \times 0.5\right) \\ &\sim N(3.759, 0.02) \end{aligned}$$

- Note that $E[S_T] = S_0 e^{\mu T}$ and $Var[S_T] = S_0^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right)$.

The Distribution of the Rate of Return

- Let R be the continuously compounded rate of return per annum realized between times 0 and T. Then, by definition,

$$S_T = S_0 e^{RT} \tag{5}$$

and so

$$R = \frac{1}{T} \ln \frac{S_T}{S_0} \tag{6}$$

- Hence,

$$R \sim N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \tag{7}$$

The Distribution of the Rate of Return: Example

- Consider a stock with an expected return of 17% per annum, and a volatility of 20 % per annum.
- The continuously compounded rate of return per annum realized over 3 years is distributed $N\left(0.17 - \frac{0.20^2}{2}, \frac{0.20^2}{3}\right)$.

Assumptions of BSM

- The stock price S follows

$$dS = \mu S dt + \sigma S dz \tag{8}$$

- Short selling is allowed.
- No transaction costs or taxes.
- Securities are perfectly divisible.
- No dividend during the life of derivatives.
- No arbitrage opportunities.
- Security trading is continuous.
- The risk-free rate of interest, r, is constant and the same for all maturities.

Black-Scholes-Merton PDE: Derivation

- The main ideas are
  1. to apply Ito's lemma on the option price process as a function of stock price,
  2. and to use no arbitrage condition to set the price.
- Let $S(t)$ be the stock price at time t, and T be the maturity date of the derivative. That is, the time to maturity is $T - t$.
- Let $f(S, t)$ be the price of the derivative at time t, and it depends on the stock price.
- Using Ito's lemma, we can show that

$$df = \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial f}{\partial S}\sigma S dz \tag{9}$$

where $dz = \epsilon\sqrt{dt}$.

Black-Scholes-Merton PDE: Risk-Free Portfolio

- We can now form a risk-free portfolio by
  1. short on the derivative: -1,
  2. long the stock: $\frac{\partial f}{\partial S}$.
- The value of the portfolio is

$$\Pi = -f + \frac{\partial f}{\partial S} S \tag{10}$$

- Hence, we can show that

$$\begin{aligned} d\Pi &= -df + \frac{\partial f}{\partial S} dS = -\left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right] dt \\ &- \frac{\partial f}{\partial S}\sigma S dz + \frac{\partial f}{\partial S}\mu S dt + \frac{\partial f}{\partial S}\sigma S dz \\ &= -\left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right] dt \end{aligned}$$

- Since this portfolio value does not have dz term, it is risk free.

Black-Scholes-Merton PDE: No-Arbitrage

- Using the no-arbitrage condition, $d\Pi = r\Pi dt$,

$$-\left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right] dt = r\left(-f + \frac{\partial f}{\partial S}S\right) dt \tag{11}$$

- Hence, we get the Black-Scholes-Merton (stochastic) Partial Differential Equation (PDE)

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{12}$$

Black-Scholes-Merton PDE: Boundary Condition

- The Black-Scholes-Merton (stochastic) Partial Differential Equation (PDE)

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{13}$$

  has many solutions, each of which is a tradeable derivative (satisfying the assumptions above) that will not create an arbitrage opportunity.
- To get the price of a specific derivative, we need to specify a boundary condition. For example,
  1. European call option: $f = \max\{S - K, 0\}$ when $t = T$,
  2. European put option: $f = \max\{K - S, 0\}$ when $t = T$

Black-Scholes-Merton PDE: Example

- In principle, we can try to solve an PDE with a boundary condition using mathematical program. But we will not do it in this class yet.
- We will just check if price of a derivative satisfies the PDE or not.
- Example: consider a forward price

$$f = S - Ke^{-r(T-t)} \tag{14}$$

- Ask students to show that it satisfies the BSM PDE.

Black-Scholes-Merton PDE: Arbitrage Example

- Now what if a price function does not satisfy the BSM PDE?
- Then there must be an arbitrage opportunity.
- Example: consider $f = e^S$. It does not satisfies the BSM PDE.
- Consider the value of the risk-free portfolio in this case.

$$d\Pi = -\left[0 + \frac{1}{2}e^S\sigma^2 S^2\right] dt \tag{15}$$

- There is an arbitrage opportunity if

$$-\frac{1}{2}e^S\sigma^2 S^2 - r(e^S - e^S S) > 0 \Rightarrow -\frac{1}{2}\sigma^2 S^2 + rS - r > 0 \tag{16}$$

- For example, if $\sigma = 0.2, r = 0.1$, you can show that you can arbitrage using this risk-free portfolio when $1.382 \leq S \leq 3.618$.

Risk Neutral Evaluation

- Note that the BSM PDE does not depend on μ. On the other hand, μ (expected return required by investors) is the only part of stock price that depends on risk preferences.
- That is, BSM PDE is independent of risk preferences. hence, we can use any set of risk preferences to evaluate the value of derivatives.
- Hence, we will use risk neutral evaluation, which implies that
  1. $\mu = r$.

Risk Neutral Evaluation: Example

- Consider a forward contract on a non-dividend paying stock.
- Let K be the delivery price,a dn T is maturity.
- The value in risk neutral world is

$$
\begin{aligned}
f &= e^{-rT}E(S_T - K) = e^{-rT}E(S_T) - e^{-rT}K \\
&= e^{-rT}S_0e^{rT} - e^{-rT}K = S_0 - e^{-rT}K
\end{aligned}
$$

- Note that we use BSM model indirectly here. We use it to ensure that we can use risk neutral valuation ($\mu = r$).

Black-Scholes-Merton Pricing Formulas

- We can solve the PDE for the pricing formulas. But it is beyond the scope of this class.
- We will use risk neutral valuation to get the pricing formulas at the end of the lecture.
- Let $\Phi(x)$ is the cumulative distribution function of the standard normal.

Black-Scholes-Merton Pricing Formulas

- The Black-Scholes-Merton pricing formulas at time t for European call and put options with maturity at T , respectively,

$$\begin{aligned} c &= S_0\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) && (17) \\ p &= Ke^{-r(T-t)}\Phi(-d_2) - S_0\Phi(-d_1) && (18) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} && (19) \\ d_2 &= \frac{\ln\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} && (20) \end{aligned}$$

Black-Scholes-Merton Pricing Formulas: Interpretation

- Using risk neutral evaluation, we can write (the price at the beginning)

$$c = e^{-rT} E \left[\max\{S - K, 0\} \right] \tag{21}$$

- Using the formula above we can rewrite it as

$$c = e^{-rT} \left[S_0 e^{rT} \Phi(d_1) - K \Phi(d_2) \right] \tag{22}$$

- $S_0 \Phi(d_1)$ is the expected value in risk-neutral world of a variable that is equal to S_T if $S_T - K > 0$ and to zero otherwise.
- $K \Phi(d_2)$ is the expected cost if the buyers exercise the option.

Black-Scholes-Merton Pricing Formulas: Double Check

- We will now check whether the formula can apply to simpler derivatives that we know their prices.
- First, consider a forward contract whose price is $S_0 - Ke^{-rT}$.
- A forward contract can be considered as a call option when S_0 is so large that almost certain that it will be exercised.
- When S_0 is so large, we will get d_1 and d_2 are also very large. that means $\Phi(d_1) \approx \Phi(d_2) \approx 1$.
- Therefore, the price is

$$c = S_0\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = S_0 - Ke^{-r(T-t)}$$

Black-Scholes-Merton Pricing Formulas: Double Check

- Second, consider when σ approaches zero.
- That means it is almost certain that the stock price at time T will be $e^{rT}S_0$. Hence, the payoff of the option is $\max\{e^{rT}S_0 - K, 0\}$.
- The value of the option today is $\max\{S_0 - e^{-rT}K, 0\}$.
- Now consider the BSM formula.
  1. Case I: when $S_0 > e^{-rT}K$. As $\sigma \to 0$, we can show that $d_1, d_2 \to \infty$. This we get $c = S_0 - Ke^{-r(T-t)}$.
  2. Case II: when $S_0 < e^{-rT}K$. As $\sigma \to 0$, we can show that $d_1, d_2 \to -\infty$. This case $\Phi(d_1) \approx \Phi(d_2) \approx 0$. hence, $c = 0$.

Option Pricing with Dividend

- For European options we can simply value the dividend stream as riskless part separately. Then, the BSM applies to the risky part.
- Example: Consider a European call option on a stock when there are ex-dividend dates in 2 months and in 5 months.
- The dividend is expected to be \$0.50.
- Let $S_0 = 40, K = 40, \sigma = 0.30, r = 0.09, T = 0.5.$
- We can first calculate the discounted value of the dividend stream:

$$0.50 \times^{-0.09 \times \frac{2}{12}} + 0.50 \times^{-0.09 \times \frac{5}{12}} = 0.9742 \tag{23}$$

Option Pricing with Dividend

- The effective price of risky part of the stock is $S_0 - 0.9742 = 39.0258$.
- Hence, we have

$$d_1 = 0.2020, d_2 = -0.0102 \Rightarrow \Phi(d_1) = 0.5800, \Phi(d_2) = 0.4959.$$

- Using the BSM formula, we can get

$$c = 39.0258 \times 0.5800 - 40 \times e^{-0.09 \times 0.50} \times 0.4959 = 3.67$$

Estimating Volatility from Historical Data: Simple Method

- Since we assume that the process of $u_t = \ln \frac{S_t}{S_{t-1}}$ is distributed normal $N\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right)$, we can simply calculate SD of this process using

$$s = \sqrt{\frac{1}{n-1}\sum_{t=1}^{T}(u_t - \bar{u})^2} \tag{24}$$

where $t = 0, 1, \ldots, T$.

- This SD is $\sigma\sqrt{t}$ where t is the length of time of each period of the data in unit of the length of time we want to have. Hence, we can estimate the volatility using

$$\hat{\sigma} = \frac{s}{t} \tag{25}$$

Estimating Volatility from Historical Data: Example

- Consider data of 21 consecutive trading days of a stock price. So, $T = 20$.
- Let

$$\sum_{t=1}^{T} u_i = 0.09531, \sum_{t=1}^{T} u_i^2 = 0.00326$$

- Hence,

$$s = \sqrt{\frac{0.00326}{19} - \frac{0.09531^2}{20 \times 19}} = 0.01216$$

- Given that there are 252 trading days per year, $t = \frac{1}{252}$. Hence, the volatility per annum is $0.01216 \times \sqrt{252} = 0.193$.

Implied Volatility

- We can also estimate volatility using option prices already traded in the market.
- In this case, we do not need to know the process of S directly since BSM formula do not need μ and σ is our unknown anyway.
- Example: there is a European call option on a non-dividend-paying stock whose value is $c = 1.875$. Let $S_0 = 21, K = 20, r = 0.10, T = 0.25$.
- What is σ?
- We can get σ using BSM formula (17) by substituting $S_0 = 21, K = 20, r = 0.10, T = 0.25$ into the equation and solve for σ that gives $c = 1.875$.
- You can use Matlab or even Excel to solve for this.